MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Independent trials

Next: ASV 2.4-2.5

Week 3:

- Homework 3 due Friday, February 9
- Midterm 1 (Wednesday, February 1, lectures 1-8)

Properties of the CDF

$F_X(r) = P(X \le r)$

(1) Monotone increasing: $s < t$, then $F_X(s) \le F_X(t)$

(2) $\lim_{r \to -\infty} F_X(r) = 0$, $\lim_{r \to +\infty} F_X(r) = 1$

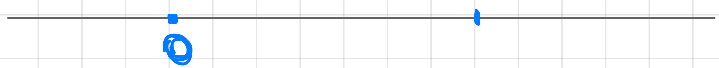
(3) The function F_X is right-continuous:

$$\lim_{t \to r_+} F_X(t) = F_X(r)$$

Corollary: If X is a continuous random variable, F_X is a continuous function

Example Shoot an arrow at a circular target of radius 1 (choose point from unit disk uniformly at random)

$$F_X(r) = \begin{cases} 0, & r \le 0 \\ r^2, & 0 \le r \le 1 \\ 1, & r \ge 1 \end{cases}$$



Cumulative distribution function (CDF)

Summary: For any random variable X, $F_X(r) = P(X \leq r)$

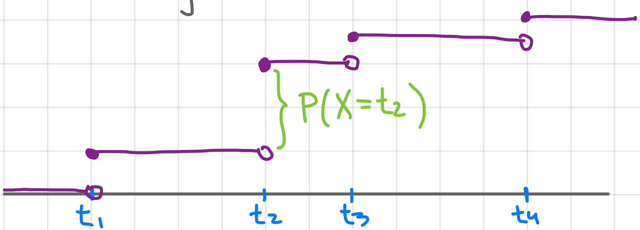
(1) Monotone increasing: $s \leq t \Rightarrow F_X(s) \leq F_X(t)$

(2) $\lim_{r \to -\infty} F_X(r) = 0$, $\lim_{r \to +\infty} F_X(r) = 1$

(3) Right-continuous: $\lim_{t \to r_+} F_X(t) = F_X(r)$

Discrete random variable

Finite or countable set of values with $t_1, t_2, \ldots$, $P(X = t_j) > 0$ and $\sum_j P(X = t_j) = 1$

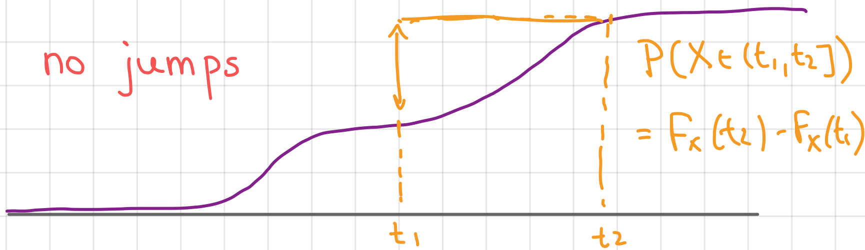


Continuous random variable

For each real number t, $P(X=t)=0$

Because (1) and (3) this implies that F_X is continuous

no jumps



Densities (PDF)

Some continuous random variables have probability densities. This is the infinitesimal version of the probability mass function.

X discrete, $X \in \{t_1, t_2, \dots\}$

$p_X(t) = P(X=t)$

probability mass function

X continuous

$P(X=t) = 0$ for all $t \in \mathbb{R}$

Densities (PDF)

Example Shoot an arrow at a circular target of radius 1.

X = distance from center

$$F_X(r) = \begin{cases} 0, & r \le 0 \\ r^2, & 0 \le r \le 1 \\ 1, & r \ge 1 \end{cases}$$

PDF: existence

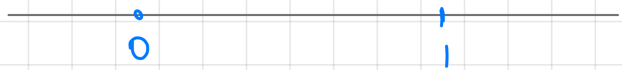
Thm: If F_X is continuous and (piecewise) differentiable, then X has density

Proof: Follows from FTC $\blacksquare$

Example Let X = random number chosen uniformly on $[0,1]$

We have seen that in this case $P(X \in [s,t]) = t-s$, $0 \leq s < t \leq 1$

$$F_X(r) = P(X \leq r) = \begin{cases} \end{cases}$$



$$f_X(r) =$$

PDF

Example Let $f(t) = \begin{cases} c\sqrt{1-t^2}, & |t| \le 1 \\ 0, & \text{otherwise} \end{cases}$, $c > 0$

Q: (When) Is $f(t)$ a PDF of some random variable?

- $f \ge 0$
- $1 = \int_{-\infty}^{+\infty} f(t)\,dt =$

f is a PDF

Question

Your car is in a minor accident. The damage repair cost is a random number between 100 and 1500 dollars. Your insurance deductible is 500 dollars.

Z = your out of pocket expenses

Question: The random variable Z is

(a) continuous

(b) discrete

(c) neither

(d) both

Independent random variables

A collection $X_1, X_2, \dots, X_n$ of random variables defined on the same sample space are independent if for any $B_1, B_2, \dots, B_n \subset \mathbb{R}$, the events

i.e.,

Special case: if X_j are discrete random variables, it suffices to check the simpler condition

for any real numbers $t_1, t_2, \dots, t_n$

Example Let $X_1, X_2, \dots, X_n$ be fair coin tosses, $H \sim 1$, $T \sim 0$

$P(X_1 = t_1, X_2 = t_2, \dots, X_n = t_n) =$ $= P(X_1 = t_1) \cdots P(X_n = t_n)$

Bernoulli distribution

Experiments can have numerical observables, but sometimes you only observe whether there is

success or failure

We model this with a random variable X taking value 1 with probability p, and value 0 with probability $1-p$

In practice, we usually repeat the experiment many times, making sure to use the same setup each trial. The previous trials do not influence the future ones.

Independent trials. Binomial distribution

Let $X_1, X_2, \dots, X_n$ be independent Ber(p) random variables

E.g. $P(X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 0, X_5 = 0, X_6 = 0)$

$=$

$=$

Run n independent trials, each with success probability p,

$$X_1, X_2, \dots, X_n \sim \text{Ber}(p)$$

Let $S_n = \#$ successful trials

What is the distribution of S_n?

$P(S_n = k) = P(\{\text{exactly } k \text{ of the } n \text{ trials are successful}\})$

$0 \le k \le n$

$=$

If , # of Heads in n tosses

Independent trials. Binomial distribution

Example Roll a fair die 10 times. What is the probability that 6 comes up at least 3 times?

success

$X_1, X_2, \dots, X_{10} \sim$

$P(S_{10} \geq 3) =$

$=$

$=$

What is the probability that no 6 is rolled in the 10 rolls?

$P(S_{10} = 0) =$

First success time. Geometric distribution

Keep rolling. Let N denote the first roll where a 6 appears. N is a random variable.

What is the distribution of N?

N = first success in repeated independent trials (success rate p).

Model trials with (unlimited number of) independent Ber(p)'s

$$\{N=k\} = \{X_1=0, X_2=0, \dots, X_{k-1}=0, X_k=1\}$$

$$P(N=k) =$$

$$=$$

Geometric Distribution Geom(p) on $\{1,2,3,\dots\}$. Is it?

Rare events. Poisson distribution

Let $\lambda > 0$ and let X be a r.v. taking values in $\{0, 1, 2, \dots\}$.
X has Poisson distribution with parameter λ if

$$P(X=k) = \qquad\qquad \text{for } k \in \{0,1,2,\dots\}$$

We write

Poisson distribution describes the probability that a "rare" event occurs k times after repeating the experiment (independent trials) "many" times.

Is this a probability distribution?

$P(X=k) \geq 0$,

λ gives the "expected number" of occurrances

Poisson distribution. Example

Observation: between 1875 and 1894 (20 years) in 14 units of Prussian army there were 196 deaths from horse kicks, distributed in the following way

# deaths per unit per year, k	# unit-years with k deaths	empirical probability	$P(X=k)$
0	144		
1	91		
2	32		
3	11		
4	2		
5+	0		

total 280

Let is "expected number" of death per unit