MATH 180A (Lecture A00)

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Today: Independent trials

Next: ASV 3.3

Week 5:

- Homework 3 due Friday, February 10

Independent random variables

A collection $X_1, X_2, \dots, X_n$ of random variables defined on the same sample space are independent if for any $B_1, B_2, \dots, B_n \subset \mathbb{R}$, the events

$$\{X_1 \in B_1\}, \{X_2 \in B_2\}, \dots, \{X_n \in B_n\} \text{ are independent}$$

i.e., $$P(X_1 \in B_1, X_2 \in B_2, \dots, X_n \in B_n) = P(X_1 \in B_1) \cdot P(X_2 \in B_2) \cdots P(X_n \in B_n)$$

Special case: if X_j are discrete random variables, it suffices to check the simpler condition

for any real numbers $t_1, t_2, \dots, t_n$

$$P(X_1 = t_1, X_2 = t_2, \dots, X_n = t_n) = P(X_1 = t_1) \cdot P(X_2 = t_2) \cdots P(X_n = t_n)$$

<u>Example</u> Let $X_1, X_2, \dots, X_n$ be fair coin tosses, $H \sim 1$, $T \sim 0$

$$P(X_1 = t_1, X_2 = t_2, \dots, X_n = t_n) = \frac{1}{2^n} = \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}}_{n \text{ times}} = P(X_1 = t_1) \cdots P(X_n = t_n)$$

Bernoulli distribution

Experiments can have numerical observables, but sometimes you only observe whether there is

success or failure

We model this with a random variable X taking value 1 with probability p, and value 0 with probability $1-p$

$$X \sim Ber(p) \quad (\text{Bernoulli})$$

In practice, we usually repeat the experiment many times, making sure to use the same setup each trial. The previous trials do not influence the future ones.

Independent trials. Binomial distribution

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Let $X_1, X_2, \dots, X_n$ be independent Ber(p) random variables

E.g. $P(X_1=0, X_2=1, X_3=1, X_4=0, X_5=0, X_6=0)$

$= P(X_1=0)\,P(X_2=1)\,P(X_3=1)\,P(X_4=0)\,P(X_5=0)\,P(X_6=0)$

$= (1-p)\cdot p\cdot p\,(1-p)(1-p)(1-p) = p^2(1-p)^4$

Run n independent trials, each with success probability p,

$$X_1, X_2, \cdots, X_n \sim \mathrm{Ber}(p)$$

Let $S_n = \#$ successful trials $= X_1 + X_2 + X_3 + \cdots + X_n$

What is the distribution of S_n?

$P(S_n = k) = P(\{\text{exactly } k \text{ of the } n \text{ trials are successful}\})$

$0 \le k \le n$

$= \binom{n}{k} p^k (1-p)^{n-k}$ Binomial distribution Bin(n, p)

If $p=\frac{1}{2}$, $1-p=\frac{1}{2}$, $P(S_n=k)=\binom{n}{k}\frac{1}{2^n}$, # of Heads in n tosses $\sim \mathrm{Bin}(n, \frac{1}{2})$

Independent trials. Binomial distribution

Example Roll a fair die 10 times. What is the probability that 6 comes up at least 3 times?

success

$$X_1, X_2, \dots, X_{10} \sim \text{Ber}\left(\frac{1}{6}\right)$$

$$S_{10} = X_1 + X_2 + \dots + X_{10} \sim \text{Bin}\left(10, \frac{1}{6}\right)$$

$$P(S_{10} \geq 3) = \sum_{k \geq 3} P(S_{10} = k) = 1 - P(S_{10} < 3)$$

$$= 1 - P(S_{10} = 0) - P(S_{10} = 1) - P(S_{10} = 2)$$

$$= 1 - \binom{10}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10} - \binom{10}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^9 - \binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8$$

What is the probability that no 6 is rolled in the 10 rolls?

$$P(S_{10} = 0) = \binom{10}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{10} = \left(\frac{5}{6}\right)^{10}$$

First success time. Geometric distribution

$|\alpha|<1$, $\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}$

Keep rolling. Let N denote the first roll where a 6 appears. N is a random variable.

What is the distribution of N?

N = first success in repeated independent trials (success rate p).

Model trials with (unlimited number of) independent Ber(p)'s

$X_1, X_2, X_3, \ldots$ $\qquad N \in \{1, 2, 3, 4, \ldots\}$

$$\{N=k\} = \{X_1=0, X_2=0, \ldots, X_{k-1}=0, X_k=1\}$$

$$P(N=k) \overset{\text{indep.}}{=} P(X_1=0)\, P(X_2=0) \cdots P(X_{k-1}=0)\, P(X_k=1)$$

$$= (1-p)^{k-1}\, p$$

Geometric Distribution Geom(p) on $\{1,2,3,\ldots\}$. Is it?

$$\sum_{k=1}^{\infty} P(N=k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \sum_{k=1}^{\infty} (1-p)^{k-1=\ell} = p \sum_{\ell=0}^{\infty} (1-p)^{\ell} = p \cdot \frac{1}{1-(1-p)} = 1.$$

Rare events. Poisson distribution

$\forall x \in \mathbb{R} \quad \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$

Let $\lambda > 0$ and let X be a r.v. taking values in $\{0,1,2,\dots\}$.

X has Poisson distribution with parameter λ if

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{for } k \in \{0,1,2,\dots\}$$

We write $X \sim \text{Pois}(\lambda)$

Poisson distribution describes the probability that a "rare" event occurs k times after repeating the experiment (independent trials) "many" times.

Is this a probability distribution?

$$P(X=k) \geq 0, \quad \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

λ gives the "expected number" of occurrances