MATH 180A (Lecture A00)

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Today: Gaussian distribution

Next: ASV 4.1

Week 6:

- Homework 4 due Friday, February 17

Variance

Definition The variance of a random variable X is

$$\mathrm{Var}(X) = E\left(\left(X - E(X)\right)^2\right)$$

Proposition. Let X be a random variable. Then

$$\mathrm{Var}(X) = E(X^2) - \left(E(X)\right)^2$$

- The square root of the variance is called

standard deviation $\sigma(X) = \sqrt{\mathrm{Var}(X)}$

Variance

Variance is a measure of how "spread out from the mean" the distribution is.

Proposition Let X be a random variable with finite expectation $E(X)=\mu$. Then

Proof ($\Leftarrow$) Exercise

($\Rightarrow$) (Assume X is discrete).

$0 = \mathrm{Var}(X) =$ $\Rightarrow$ For all t,

For all t, either or , so if

therefore,

Expectation and variance of $aX+b$

Let X be a random variable, and let $a, b \in \mathbb{R}$. Then

(i) $E(aX+b) =$

(ii) $Var(aX+b) =$ if $E(X)$ and $Var(X)$ exist

Proof (i) $\leftarrow$ homework

(ii) $Var(aX+b) =$

$=$

$=$

$=$

$=$

Variance of geometric distribution

Let $X \sim Geom(p)$. We know that $E(X) = \frac{1}{p}$

$$E(X^2) = \sum_{k=1}^{\infty} k^2 P(X=k) = \sum_{k=1}^{\infty} k^2 \cdot p(1-p)^{k-1} =$$

$$=$$

$l = k-1 \quad =$

$$=$$

$$=$$

Random variables. Summary

Discrete	Continuous
Finite/countable set of possible values, $\sum_t P(X=t)=1$	Uncountable set of possible values, $\forall t\in\mathbb{R}$ $P(X=t)=0$
PMF: $p_X(t)=P(X=t)$ $P(X\in B)=\sum_{t\in B} p_X(t)$	PDF: $f_X:\mathbb{R}\to\mathbb{R}$ $P(X\in B)=\int_B f_X(t)\,dt$
CDF F_X is a step function	CDF F_X is a continuous function
Expectation: $E(X)=\sum_t t\,P(X=t)$	Expectation: $E(X)=\int_{\mathbb{R}} t\,f_X(t)\,dt$
$E(g(X))=\sum_t g(t)\,P(X=t)$	$E(g(X))=\int_{\mathbb{R}} g(t)\,f_X(t)\,dt$
Relation between CDF and PMF: magnitude of jump of F_X at t is $P(X=t)$	Relation between CDF and PDF: $f_X(t)=F_X'(t)$ on the intervals where F_X is differentiable

Random variables. Summary

F_X is (i) nondecreasing, (ii) right-continuous

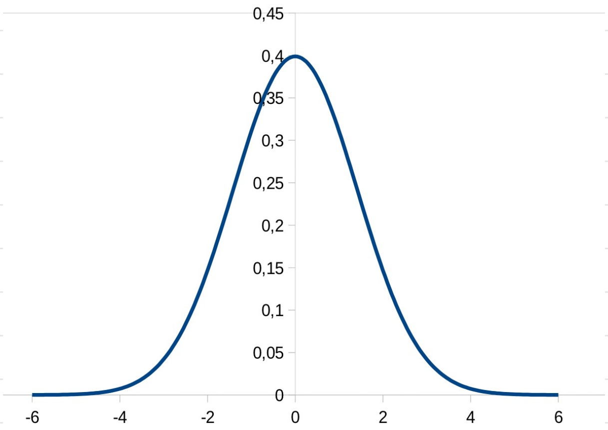
$$\text{(iii)} \lim_{t\to-\infty} F_X(t) = 0, \quad \lim_{t\to+\infty} F_X(t) = 1$$

Variance: $\quad Var(X) = E\left(\left(X - E(X)\right)^2\right) = E(X^2) - \left(E(X)\right)^2$

$$E(aX+b) = aE(X)+b, \qquad Var(aX+b) = a^2 Var(X)$$

Gaussian (normal) distribution

<u>Def</u> Random variable Z has the standard normal (Gaussian) distribution if the PDF of Z is given by



Notation: $Z \sim \mathcal{N}(0,1)$

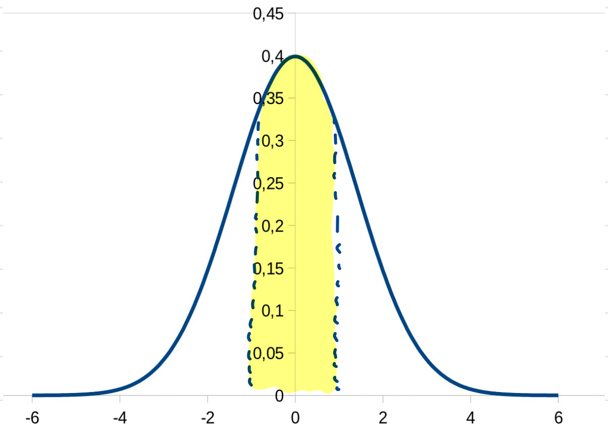
Is $\varphi(t)$ indeed a PDF?

- $\varphi(t) \geq 0$
- $\displaystyle\int_{-\infty}^{+\infty} \varphi(t)\,dt = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt \overset{?}{=} 1$

CDF of $N(0,1)$

Suppose $X \sim N(0,1)$. What is $P(|X| \leq 1)$?

$P(-1 \leq X \leq 1)$



Cannot use the polar coordinate trick.

$\Phi(x) :=$ - CDF of $X \sim N(0,1)$

- no simple explicit formula
- table of values of $\Phi(x)$ (for $x \geq 0$)

Normal table of values (Appendix E in textbook)

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389

This table gives $P(Z \leq z)$ where $Z \sim N(0,1)$, $z = x_i + y_j$

<u>Example</u> $\Phi(0.91) = P(Z \leq 0.91) =$

<u>Fact</u>:

$P(Z > 0.24) =$

$P(-0.28 < Z < 0.59) =$

Normal table of values

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
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Exercise Let $Z \sim \mathcal{N}(0,1)$

Find $x_0 \in \mathbb{R}$ such that $P(|Z| > x_0) \approx 0.704$

$P(|Z| > x_0) =$

Mean and variance of $X \sim N(0,1)$

$$E(X) = \int_{-\infty}^{+\infty} t\, f_X(t)\, dt =$$

$$Var(X) = E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t^2 e^{-\frac{t^2}{2}} dt$$

General normal distribution $N(\mu, \sigma^2)$

Def Let $\mu \in \mathbb{R}$ and $\sigma > 0$. Random variable X has normal (Gaussian) distribution with mean μ and variance σ^2 if the PDF of X is given by

$$f_X(x) =$$

We write

Using the density we can compute

$$E(X) = \qquad , \quad Var(X) =$$

"Gaussian distribution" = family of distributions

Relation between $X \sim N(\mu, \sigma)$ and $Z \sim N(0,1)$

Proposition Let $X \sim N(\mu, \sigma^2)$, $a \neq 0$, $b \in \mathbb{R}$.

Then

Using this proposition any Gaussian random variable can be written as a shifted and rescaled standart normal.

E.g., if $\sigma > 0$, $\mu \in \mathbb{R}$ and $Z \sim N(0,1)$, then

If $X \sim N(\mu, \sigma^2)$, then $E(X) =$; $Var(X) =$

If $X \sim N(\mu, \sigma^2)$, then

Example

Let $X \sim N(-3, 4)$

Find $P(X < 0.91)$; $P(X > 0.82)$; $P(-0.24 < X < 0.88)$

If $X \sim N(-3, 4)$, then , so

$P(X < 0.91) =$

$P(-0.24 < X < 0.88) =$

$=$