# MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

## Today: Normal approximation of Bin(n,p)

## Next: ASV 4.2-4.3

Week 7:

- no homework
- Midterm 2 on Wednesday, March 1 (lectures 8-18)
- Homework 5 due Friday, March 3

"This section is among the most significant ones in the text, and one whose message should stay with you long after reading this book. The idea here is foundational to just all human activity…"

*Introduction to Probability* D. Anderson, T. Seppäläinen, B. Valkó

## The message:

If we have independent and identically distributed random variables $X_1, X_2, \dots, X_n$ with $E(X_1)=\mu$, $Var(X_1)=\sigma^2$, then for any $a<b$

$$\lim_{n\to\infty} P\left(a \le \frac{X_1+X_2+\dots+X_n-\mu n}{\sqrt{n}\,\sigma} \le b\right) = \int_a^b \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\,dt$$

$$= \Phi(b) - \Phi(a)$$

CENTRAL LIMIT THEOREM

Today: $X_1 \sim Ber(p)$; Last lecture: general case

## CLT for Bernoulli distribution (approximation of Bin)

If $X_i \sim Ber(p)$ are independent, then $X_1 + \cdots + X_n \sim Bin(n,p)$

$E(X_1) = p$, $Var(X_1) = p(1-p)$

### CLT for Bernoulli distribution:

Let $S_n \sim Bin(n,p)$, let $a<b$. Then

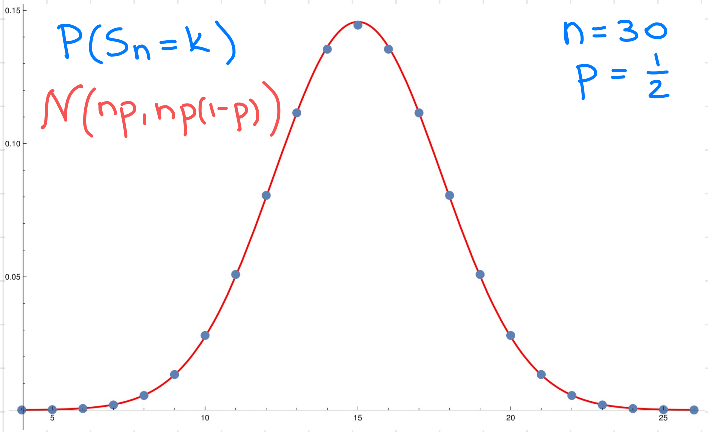
$$\lim_{n \to \infty} P\left(a < \frac{S_n - np}{\sqrt{np(1-p)}} < b\right) = \Phi(b) - \Phi(a)$$

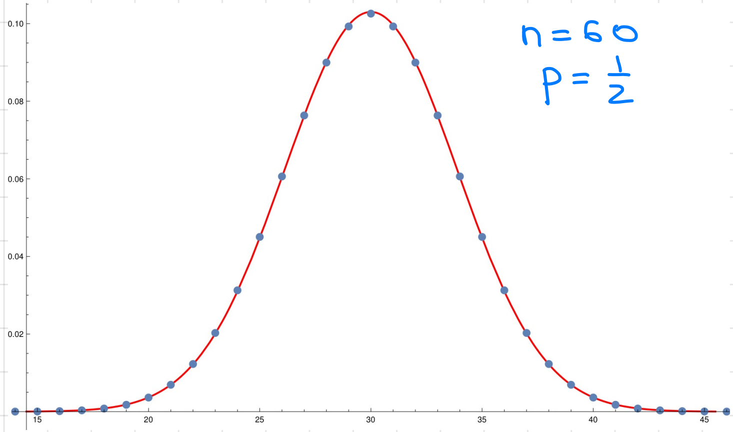
We can rewrite $(*)$ using $\overline{S_n} := \frac{S_n}{n}$

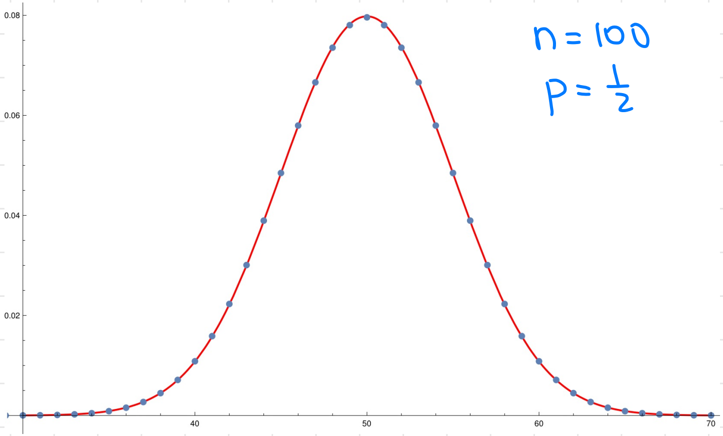
$$P\left(a \cdot \frac{\sqrt{p(1-p)}}{\sqrt{n}} < \overline{S_n} - p < b \cdot \frac{\sqrt{p(1-p)}}{\sqrt{n}}\right) \approx \Phi(b) - \Phi(a)$$

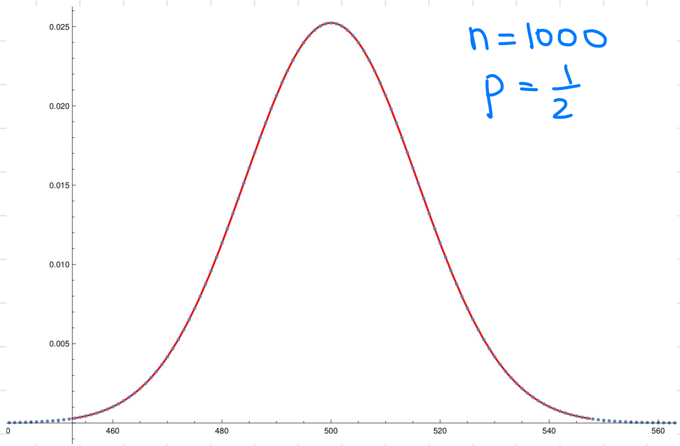
# CLT, approximation of Binomial distribution

## Some numerics



# Normal approximation. 3-sigma rule

We use the approximation of $\mathrm{Bin}(n,p)$ by the normal distribution if $np(1-p) > 10$

In this case we can take

$$P\left(a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b\right) \approx \Phi(b) - \Phi(a)$$

| Z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9924 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9958 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |

In particular, in this case

- $P\left(|S_n - np| < \sqrt{np(1-p)}\right) \approx \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.68$
- $P\left(|S_n - np| < 2\sqrt{np(1-p)}\right) \approx \Phi(2) - \Phi(-2) = 2\Phi(2) - 1 = 0.95$
- $P\left(|S_n - np| < 3\sqrt{\overbrace{np(1-p)}^{\approx 6}}\right) \approx \Phi(3) - \Phi(-3) = 2\Phi(3) - 1 = 0.99$

# CLT. Examples

| Z | 0.00 | 0.01 | 0.02 |
|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 |

Flipping a fair coin 10000 times

$X$ = number of tails

Find (approximately) $P(4950 \le X \le 5050)$

$X \sim Bin(10\,000, \frac{1}{2})$, $\quad n \cdot p(1-p) = 10000 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2500 > 10$

$E(X) = np = 5000$

$\sigma(X) = \sqrt{np(1-p)} = \sqrt{10000 \cdot \frac{1}{2} \cdot \frac{1}{2}} = \sqrt{2500} = 50$

$$P(4950 \le X \le 5050) = P\left(\frac{4950-5000}{50} \le \frac{X-5000}{50} \le \frac{5050-5000}{50}\right)$$

$$= P\left(-1 \le \frac{X-5000}{50} \le 1\right) \overset{CLT}{\approx} \Phi(1) - \Phi(-1) \approx 0.6826$$

# CLT. Examples

You win \$9 with probability $\frac{1}{20}$, lose \$1 with prob. $\frac{19}{20}$

Approximate the probability that you lose < 100 \$ after 400 games.

Denote by $X$ the number of wins after 400 games

$X \sim Bin(400, \frac{1}{20})$. $\quad n \cdot p \cdot (1-p) = 400 \cdot \frac{1}{20} \cdot \frac{19}{20} = 19 > 10$

approximate $X$ by $\mathcal{N}$

Total winnings after 400 games: $9 \cdot X - 1 \cdot (400 - X)$

We have to compute

$$P(9X - (400-X) > -100) = P(10X > 300) = P(X > 30)$$

$$= P\left(\frac{X-20}{\sqrt{19}} > \underbrace{\frac{30-20}{\sqrt{19}}}_{\approx 2.294}\right) \overset{CLT}{\approx} 1 - \Phi(2.294) \approx 0.0016$$

# Law of Large Numbers

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed, and let $E(X_1) = \mu \in \mathbb{R}$. Then

for any $\varepsilon > 0$

$$\lim_{n\to\infty} P\left( \left| \frac{X_1 + X_2 + \cdots + X_n}{n} - \mu \right| < \varepsilon \right) \to 1$$

In particular, for $X_1 \sim Ber(p)$

$$\lim_{n\to\infty} P\left( \left| \frac{S_n}{n} - p \right| < \varepsilon \right) = 1$$

$$\shortparallel$$

$$P\left( \left| \frac{S_n - np}{\sqrt{n\,p(1-p)}} \right| < \frac{\sqrt{n}\,\varepsilon}{\sqrt{p(1-p)}} \right) \approx \Phi\left(\sqrt{n}\,\varepsilon\right) - \Phi\left(-\sqrt{n}\,\varepsilon\right)$$

# Example