MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Normal approximation of Bin(n,p)

Next: ASV 4.2-4.3

Week 6:

- Homework 4 due Friday, February 17

"This section is among the most significant ones in the text, and one whose message should stay with you long after reading this book. The idea here is foundational to just all human activity…"

Introduction to Probability D. Anderson, T. Seppäläinen, B. Valkó

The message:

If we have independent and identically distributed random variables $X_1, X_2, \dots, X_n$ with $E(X_1) = \mu$, $Var(X_1) = \sigma^2$, then for any $a < b$

$$\lim_{n\to\infty} P\left(a \leq \frac{X_1 + X_2 + \dots + X_n - \mu n}{\sqrt{n}\,\sigma} \leq b \right) = \int_a^b \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\, dt$$

$$= \Phi(b) - \Phi(a)$$

CENTRAL LIMIT THEOREM

Today: $X_1 \sim Ber(p)$; Last lecture: general case

CLT for Bernoulli distribution (approximation of Bin)

If $X_i \sim Ber(p)$ are independent, then $X_1 + \cdots + X_n \sim Bin(n,p)$

$E(X_1) =$, $Var(X_1) =$

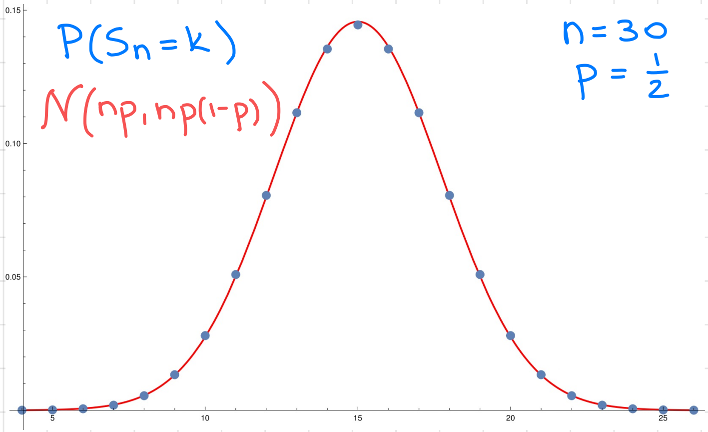
CLT for Bernoulli distribution:

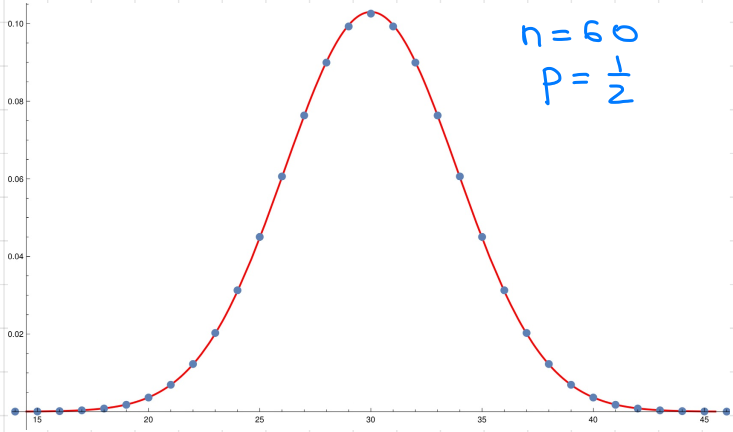
Let $S_n \sim Bin(n,p)$, let $a<b$. Then

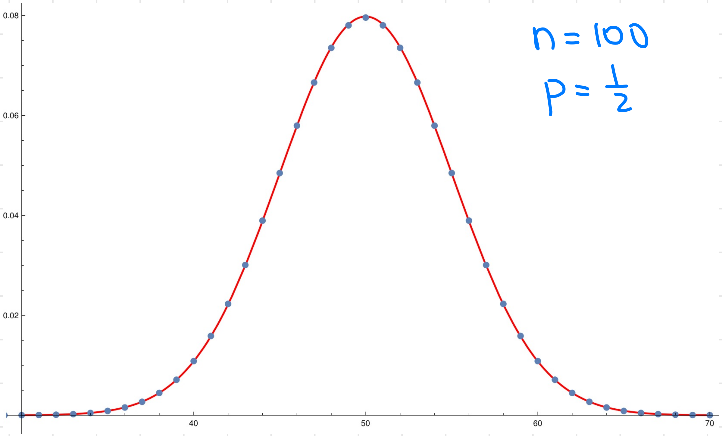
We can rewrite (*) using $\overline{S_n} := \frac{S_n}{n}$

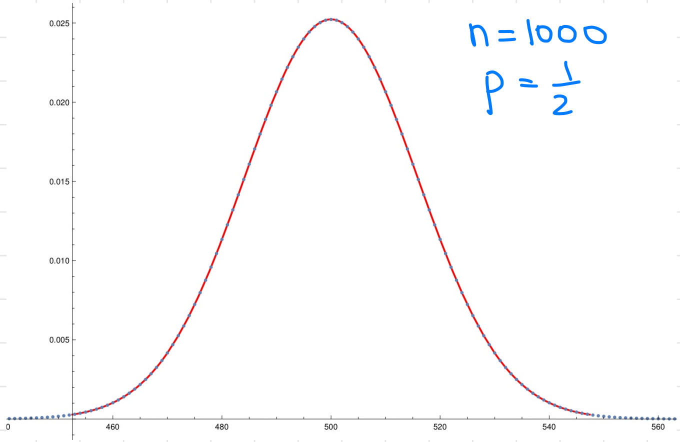
CLT, approximation of Binomial distribution

Some numerics









Normal approximation. 3-sigma rule

We use the approximation of Bin (n, p) by the normal distribution if

In this case we can take

$$P\left(a \leq \frac{S_n - np}{\sqrt{np(1-p)}} \leq b\right) \approx \Phi(b) - \Phi(a)$$

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9958	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

In particular, in this case

- $P(|S_n - np| < \qquad) \approx \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.68$
- $P(|S_n - np| < \qquad) \approx \Phi(2) - \Phi(-2) = 2\Phi(2) - 1 = 0.95$
- $P(|S_n - np| < \qquad) \approx \Phi(3) - \Phi(-3) = 2\Phi(3) - 1 = 0.99$

CLT Examples

Z	0.00	0.01	0.02
0.0	0.5000	0.5040	0.5080
0.1	0.5398	0.5438	0.5478
0.2	0.5793	0.5832	0.5871
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0.5	0.6915	0.6950	0.6985
0.6	0.7257	0.7291	0.7324
0.7	0.7580	0.7611	0.7642
0.8	0.7881	0.7910	0.7939
0.9	0.8159	0.8186	0.8212
1.0	0.8413	0.8438	0.8461
1.1	0.8643	0.8665	0.8686

Flipping a fair coin 10000 times

X = number of tails

Find (approximately) $P(4950 \leq X \leq 5050)$

$X \sim Bin(10\,000, \frac{1}{2})$

$E(X) =$

$\sigma(X) =$

$P(4950 \leq X \leq 5050) =$

CLT. Examples

You win \$9 with probability $\frac{1}{20}$, lose \$1 with prob. $\frac{19}{20}$

Approximate the probability that you lost < 100 \$ after 400 games.

Denote by X the number of wins after 400 games

$X \sim Bin(400, \frac{1}{20})$. $\quad n \cdot p \cdot (1-p) =$

Total winnings after 400 games:

We have to compute

$$P(9X - (400 - X) > -100) =$$

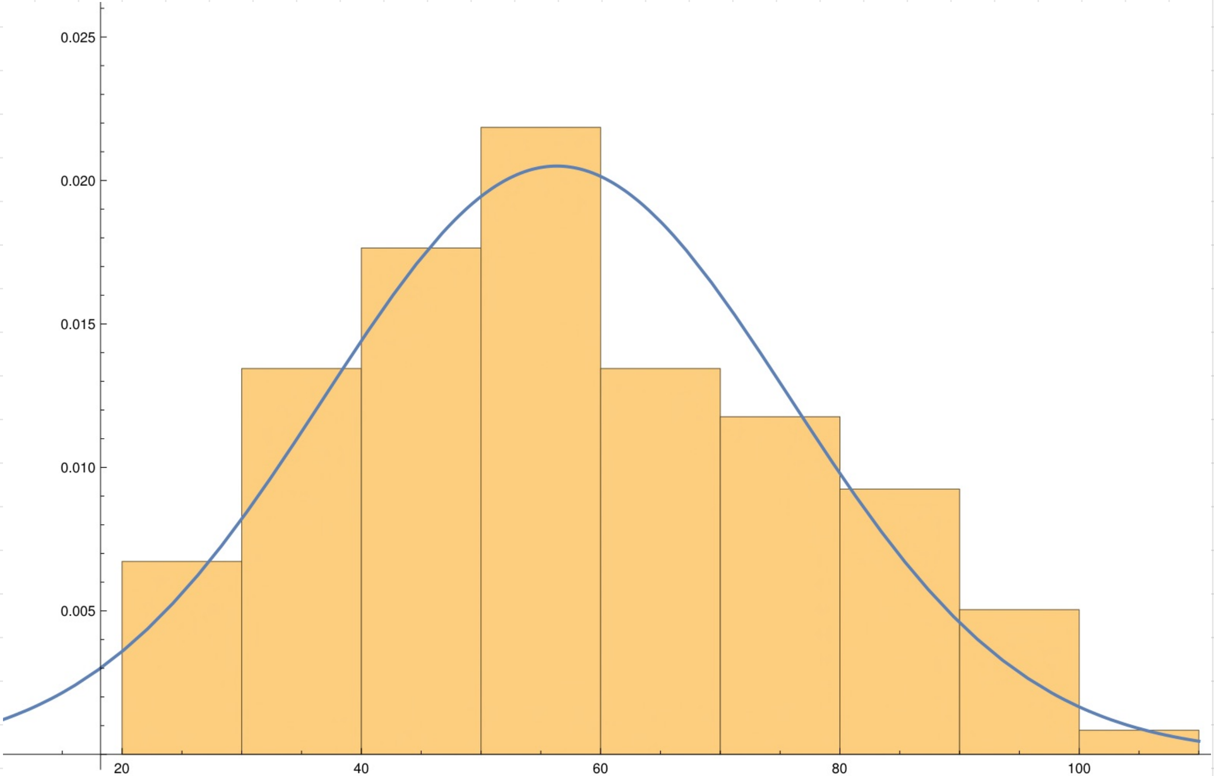
Law of Large Numbers

Let $X_1, X_2, \dots, X_n$ be independent and identically distributed, and let $E(X_1) = \mu \in \mathbb{R}$. Then

for any $\varepsilon > 0$

In particular, for $X_1 \sim Ber(p)$

Example



Summary

CLT for Bernoulli distribution:

Let $S_n \sim Bin(n,p)$, let $a<b$. Then

For the average $\overline{S_n} := \frac{S_n}{n}$

LLN for Bernoulli

Confidence intervals. Motivation

Consider n independent trials, success rate p (unknown)

S_n = number of successes after n trials, $S_n \sim \text{Bin}(n,p)$

By the LLN $\frac{S_n}{n} \to p$, $n \to \infty$

If n is big, then is close to

observable, estimate of p

Usually we do not know p, but we can get a realization of $\frac{S_n}{n}$ (flipping a coin) for finite n.

What can we say about p?

Confidence intervals. Set-up

Denote $\hat{p} := \frac{S_n}{n}$ and use the CLT for the interval $(-a, a)$

$$P\left(\frac{-a\sqrt{p(1-p)}}{\sqrt{n}} \leq \hat{p} - p \leq \frac{a\sqrt{p(1-p)}}{\sqrt{n}} \right) \approx 2\Phi(a) - 1$$

$$P\left(|\hat{p} - p| \leq \frac{a\sqrt{p(1-p)}}{\sqrt{n}} \right)$$

Questions:

1) For fixed n, find $\varepsilon > 0$ such that $P(|\hat{p}-p| \leq \varepsilon) \geq \gamma$

2) For fixed ε, find $n \in \mathbb{N}$ such that $P(|\hat{p}-p| \leq \varepsilon) \geq \gamma$

Confidence intervals

$$P(p \in [\hat{p} - \varepsilon, \hat{p} + \varepsilon]) \geq \gamma$$

Notice that $\hat{p}$ is a random variable, so the interval is random

Take some realization of

Then $[\hat{p}_* - \varepsilon, \hat{p}_* + \varepsilon]$ is the

Confidence intervals. Computations

Problem: To find an estimate of p, we use

$$P\left(|\hat{p}-p| \le \frac{a\sqrt{p(1-p)}}{\sqrt{n}} \right) \approx 2\Phi(a)-1 =: \gamma ,$$

for which ε depends on (unknown) p.

Solution: notice that

$$P\left(|\hat{p}-p| \le \frac{a\sqrt{p(1-p)}}{\sqrt{n}} \right) \approx \gamma$$

and the γ-confidence interval can be taken as

$[\hat{p}-\varepsilon, \hat{p}+\varepsilon]$ with

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