MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Exponential distribution

Next: ASV 6.1

Week 8:

- Homework 5 due Friday, March 3

Question

A fair 20-sided die is tossed 400 times.
We want to calculate the probability that a 13 came up at least 25 times. We should use

$\text{Bin}(n, p)$, $n = 400$, $p = \frac{1}{20}$

(a) Poisson approximation

(b) Normal approximation

(c) Neither

(d) Both

$np(1-p) = 19$

$np^2 = 1$

$200 \geq |P(X \in A) - P(S_n \in A)|$

Poisson: $X \sim \text{Poisson}(20)$ $\quad P(X \geq 25) \approx 15.68\%$

Normal: $Y \sim N(20, 19)$ $\quad P(Y \geq 25) \approx 12.57\%$

True value: $S_n \sim \text{Bin}(400, \frac{1}{20})$ $\quad P(S_n \geq 25) \approx 15.1\%$

Waiting for a customer

Suppose that customers arrive in a store withe rate λ customers per hour. How can we model the time until the first (or next) customer arrives?

Let X = time when the first customer arrives

$t > 0$

Additional assumptions: if the intervals are small enough, then

- only one customer can arrive per interval
- customers arrive/do not arrive for each interval independently
- $P(\text{customer arrives during } T_k) = \lambda \frac{t}{n}$

Q: What is $P(X>t) = P(\text{no customers in each } T_k)$

Exponential distribution

$$\left(1+\frac{x}{n}\right)^n \xrightarrow[n\to\infty]{} e^x$$

$$P(X>t) = P(\text{no customer in each } T_k)$$
$$= \prod_{k=1}^{n} P(\text{no customer in } T_k) = \prod_{k=1}^{n}\left(1-\frac{\lambda t}{n}\right)$$
$$= \left(1-\frac{\lambda t}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-\lambda t}$$

CDF: $$P(X\le t) = 1 - P(X>t) = \begin{cases} 1-e^{-\lambda t}, & t>0 \\ 0, & t\le 0 \end{cases}$$

PDF: $$f_X(t) = \begin{cases} \lambda e^{-\lambda t}, & t>0 \\ 0, & t\le 0 \end{cases}$$

Def. Let $\lambda>0$. We say that random variable X has exponential distribution with rate parameter λ, if

$$f_X(t) = \begin{cases} \lambda e^{-\lambda t}, & t>0 \\ 0, & t\le 0 \end{cases}, \quad \text{denote } X\sim \mathrm{Exp}(\lambda)$$

Exponential distribution

Let $X \sim Exp(\lambda)$. Then

- $E(X) = \int_0^\infty t\lambda e^{-\lambda t}\,dt = \int_0^\infty t\left(-e^{-\lambda t}\right)' dt = -t\lambda e^{-\lambda t}\Big|_0^\infty + \int_0^\infty e^{-\lambda t}\,dt = \frac{1}{\lambda}$
- $E(X^2) = \frac{2}{\lambda^2}$, so $Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$
- $P(X > t) = e^{-\lambda t}$

Exponential distribution is used to model waiting times

<u>Example</u> Length of a phone call is modeled by exponential random variable with mean 10 (minutes). What is the probability that the call takes > 8 minutes? Between 8 and 22?

X = length of the call, $X \sim Exp(\lambda)$, $E(X) = 10 = \frac{1}{\lambda}$, so $\lambda = \frac{1}{10}$. Then

$P(X > 8) = e^{-\frac{1}{10}\cdot 8} \approx 0.4493$, $P(8 < X \leq 22) = P(X > 8) - P(X > 22) = e^{-0.8} - e^{-2.2}$

Memoryless property

Proposition Let $X \sim Exp(\lambda)$, $\lambda > 0$. Then for any $s, t > 0$

$$P(X > s+t \mid X > s) = P(X > t)$$

Proof $P(X > s+t \mid X > s) = \dfrac{P(X > s+t,\ X > s)}{P(X > s)} = \dfrac{P(X > s+t)}{P(X > s)}$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

$Exp(\lambda)$ is the only continuous distribution with memoryless property.

Remark. If $N \sim Geom(p)$, then $P(N > k) = (1-p)^k$, and

$$P(N > k+\ell \mid N > k) = \frac{P(N > k+\ell)}{P(N > k)} = \frac{(1-p)^{k+\ell}}{(1-p)^k} = (1-p)^\ell = P(N > \ell)$$

Example

Animals are crossing a highway. Intervals between arriving cars have exponential distribution with mean 30 (min). Turtle needs 10 minutes to cross.

(a) What is the probability that the turtle survives?

$X =$ time until car arrives, $X \sim Exp(\lambda)$, $E(X) = 30 = \frac{1}{\lambda}$

$\lambda = \frac{1}{30}$, so $X \sim Exp(\frac{1}{30})$

$$P(X > 10) = e^{-\frac{1}{30} \cdot 10} = e^{-\frac{1}{3}} \approx 0.7165$$

(b) When the turtle starts crossing the highway, a racoon says that it has not seen a car for 5 minutes. Will this change the survival probability?

No! $\quad P(X > 5+10 \mid X > 5) = P(X > 10) \approx 0.7165$