MATH 180A (Lecture A00)

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Today: Exponential distribution

Next: ASV 6.1

Week 8:

- Midterm 2 on Wednesday, March 1 (lectures 8-18)
- Homework 5 due Friday, March 3

Question

A fair 20-sided die is tossed 400 times.
We want to calculate the probability that a 13 came up at least 25 times. We should use

(a) Poisson approximation
(b) Normal approximation
(c) Neither
(d) Both

Waiting for a customer

Suppose that customers arrive in a store withe rate λ customers per hour. How can we model the time until the first (or next) customer arrives?

Additional assumptions: if the intervals are small enough, then

- only one customer can arrive per interval
- customers arrive/do not arrive for each interval independently
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Q: What is $P(X>t)$

Exponential distribution

$P(X>t)$

CDF: $P(X \le t)$

PDF: $f_X(t) =$

<u>Def</u>. Let $\lambda > 0$. We say that random variable X has exponential distribution with rate parameter λ, if

$$f_X(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & t \le 0 \end{cases}, \quad \text{denote } X \sim \mathrm{Exp}(\lambda)$$

Exponential distribution

Let $X \sim Exp(\lambda)$. Then

- $E(X) = \int_0^\infty t \lambda e^{-\lambda t} dt = \int_0^\infty t \left(-e^{-\lambda t}\right)' dt = -t\lambda e^{-\lambda t} \Big|_0^\infty + \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}$
- $E(X^2) = \frac{2}{\lambda^2}$, so $Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$
- $P(X > t) = e^{-\lambda t}$

Exponential distribution is used to model waiting times

<u>Example</u> Length of a phone call is modeled by exponential random variable with mean 10 (minutes). What is the probability that the call takes > 8 minutes? Between 8 and 22?

Memoryless property

Proposition Let $X \sim \mathrm{Exp}(\lambda)$, $\lambda > 0$. Then for any $s, t > 0$

Proof $P(X > s+t \mid X > s)$

$\mathrm{Exp}(\lambda)$ is the only continuous distribution with memoryless property.

Remark. If $N \sim \mathrm{Geom}(p)$, then $P(N > k) = (1-p)^k$, and

$$P(N > k+\ell \mid N > k) = \frac{P(N > k+\ell)}{P(N > k)} = \frac{(1-p)^{k+\ell}}{(1-p)^k} = (1-p)^\ell = P(N > \ell)$$

Example

Animals are crossing a highway. Intervals between arriving cars have exponential distribution with mean 30 (min). Turtle needs 10 minutes to cross.

(a) What is the probability that the turtle survives?

(b) When the turtle starts crossing the highway, a racoon says that it has not seen a car for 5 minutes. Will this change the survival probability?

New section

Characterizing random variables

- PMF / PDF for discrete / continuous random variables

$$P(X \in A) = \sum_{t \in A} p_X(t) \quad , \quad P(X \in A) = \int_A f_X(t)\,dt$$

- CDF

$$F_X(t) = P(X \leq t)$$

- $E(X)$, $Var(X)$ gives partial information

- Moments $\left(E(X^k)\right)_{k \geq 1}$ (sometimes) describe uniquely the distribution

NEW TOOL: Moment generating function (MGF)

convenient when working with sums of independent RVs.

Moment generating function

Def. Let X be a random variable. Then

Examples (more in the textbook)

- $X \sim Ber(p)$, $E(e^{tX}) =$
- $X \sim Poisson(\lambda)$, $E(e^{tX}) =$
- $X \sim N(0,1)$, $E(e^{tX}) =$

Moment generating function

Examples

- $X \sim N(\mu, \sigma^2)$, use that if $Z \sim N(0,1)$, then $\sigma Z + \mu \sim N(\mu, \sigma^2)$

  $E(e^{tX}) =$

- Exercise :

  $X \sim Exp(\lambda)$

  $M_X(t) =$