MATH 180A (Lecture A00)

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Today: Moment generating function

Next: ASV 6.1

Week 9:

- Homework 6 due Friday, March 10

Characterizing random variables

- PMF / PDF for discrete / continuous random variables

$$P(X \in A) = \sum_{t \in A} p_X(t) \quad , \quad P(X \in A) = \int_A f_X(t)\,dt$$

- CDF

$$F_X(t) = P(X \leq t)$$

- $E(X)$, $Var(X)$ gives partial information

- Moments $\left(E(X^k)\right)_{k \geq 1}$ (sometimes) describe uniquely the distribution

NEW TOOL: Moment generating function (MGF)

convenient when working with sums of independent RVs.

Moment generating function

Def. Let X be a random variable. Then

$$M_X(t) := E\left(e^{tX}\right), \quad t \in \mathbb{R}$$

is called the moment generating function (MGF) of X.

Examples (more in the textbook)

- $X \sim Ber(p)$, $E(e^{tX}) = e^{t\cdot 0} \cdot P(X=0) + e^{t \cdot 1} \cdot P(X=1) = 1-p+e^{t}\cdot p$
- $X \sim Poisson(\lambda)$, $E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t\lambda)^k}{k!} = e^{-\lambda} e^{e^t \lambda} = e^{\lambda(e^t-1)}$
- $X \sim N(0,1)$, $E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2}(x^2-2tx+t^2)+\frac{t^2}{2}}}{\sqrt{2\pi}} dx = e^{\frac{t^2}{2}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-t)^2}{2}}}{\sqrt{2\pi}} dx$

$$= e^{\frac{t^2}{2}}$$

Moment generating function

Examples

- $X \sim N(\mu, \sigma^2)$, use that if $Z \sim N(0,1)$, then $\sigma Z + \mu \sim N(\mu, \sigma^2)$

$$E(e^{tX}) = E\left(e^{t(\sigma Z + \mu)}\right) = E(e^{t\sigma Z} \cdot e^{t\mu}) = e^{t\mu} E(e^{t\sigma Z})$$

$$= e^{\frac{t^2\sigma^2}{2} + t\mu}$$

$$E(e^{t\sigma Z}) = e^{\frac{t^2\sigma^2}{2}} = M_Z(t\sigma)$$

- Exercise: Let X be a discrete random variable, $P(X=1) = P(X=-1) = \frac{1}{2}$. Compute $M_X(t)$

$$M_X(t) = e^{-t} \cdot \frac{1}{2} + e^{t} \cdot \frac{1}{2} = \frac{e^t + e^{-t}}{2} = \cosh(t)$$

Moment generating function

Examples

- More generally, if X is a discrete random variable taking values $k_1, k_2, k_3, \dots$, then the MGF of X is given by
$$M_X(t) = e^{k_1 t} \cdot P(X = k_1) + e^{k_2 t} \cdot P(X = k_2) + \dots + e^{k_n t} \cdot P(X = k_n)$$

MGF is not always everywhere finite!

- Let $X \sim Exp(\lambda)$. Then
$$M_X(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx = \begin{cases} \frac{\lambda}{\lambda - t}, & \text{if } t < \lambda \\ +\infty, & \text{if } t \geq \lambda \end{cases}$$

$M_X(t)$ is finite only for $t < \lambda$

- For some distributions MGF does not exist for all $t > 0$ $(|t| > 0)$

E.g. $P(X = k) = \frac{6}{\pi^2} \frac{1}{k^2}$, $k \geq 1$

Equality in distribution

Def Let X and Y be random variables. We say that X and Y are equal in distribution if

$$P(X \in B) = P(Y \in B) \text{ for any}^{+} \; B \subset \mathbb{R}$$

We denote this by $X \overset{d}{=} Y$

In particular, if $X \overset{d}{=} Y$, then $F_X = F_Y$ (CDFs are equal), and thus PDFs/PMFs are equal.

Examples

- $X \sim \text{Unif}[0,1]$, $Y = 1 - X$

$$P(Y \le s) = P(1 - X \le s) = P(X \ge 1 - s) = \begin{cases} 0, & s \le 0 \\ s, & 0 < s \le 1 \\ 1, & s > 1 \end{cases} = P(X \le s)$$

$$F_X = F_Y, \quad P(X = Y) = P(X = 1 - X) = P\left(X = \tfrac{1}{2}\right) = 0$$

Identifying the distribution with the MGF

Theorem

Let X and Y be two random variables, let $M_X(t)$, $M_Y(t)$ be their MGFs. If there exists $\delta > 0$ s.t.

(i) $M_X(t)$ and $M_Y(t)$ are finite for $t \in (-\delta, \delta)$

(ii) $M_X(t) = M_Y(t)$ for all $t \in (-\delta, \delta)$

then $X \stackrel{d}{=} Y$, X and Y are equal in distribution

No proof

Condition (i) is necessary to be able to characterize the distribution by the MGF.

You should be able to identify the MGFs of classical distributions.

Identifying the distribution with the MGF

Discrete		Continuous	
Distribution	MGF, $M_X(t)$	Distribution	MGF, $M_X(t)$
$Ber(p)$	$1-p+pe^t$	$N(0,1)$	$e^{\frac{t^2}{2}}$
$Pois(\lambda)$	$e^{\lambda(e^t-1)}$	$N(\mu,\sigma^2)$	$e^{t\mu+\frac{\sigma^2t^2}{2}}$
$Geom(p)$	?	$Exp(\lambda)$	$\begin{cases} \frac{\lambda}{\lambda-t}, & t<\lambda \\ \infty, & t\geq\lambda \end{cases}$
$P(X=k)=p_k$	$\sum_k e^{kt}p_k$		

Examples

- If $M_X(t)=\frac{1+2e^t}{3}$, then $X\sim Ber(\frac{2}{3})$
- If $M_X(t)=\frac{e^{-10t}}{3}+\frac{2}{3}$, then $P(X=-10)=\frac{1}{3}$, $P(X=0)=\frac{2}{3}$
- If $M_X(t)=e^{5t}$, then $P(X=5)=1$
- If $M_X(t)=\exp\left(\frac{(5t+1)^2}{2}-\frac{1}{2}\right)=e^{\frac{25t^2}{2}+5t}$, then $X\sim N(5,25)$