MATH 180A (Lecture A00)

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Today: Joint distribution

Next: ASV 8.1

Week 9:

- Homework 6 due Friday, March 10

Computing moments using MGF

Differentiate $M_X(t) = E(e^{tX})$ w.r.t. t

Differentiate again

More generally

Thm. If $M_X(t)$ is bounded around $t=0$, then

No proof.

Alternatively,

Computing moments using MGF. Examples

- $P(X=1) = P(X=-1) = \frac{1}{2}$, $M_X(t) =$

- $X \sim N(0,1)$, $M_X(t) = e^{\frac{t^2}{2}}$

$M_X(t) = e^{t^2/2} =$

Distribution of a function of a random variable

Let X be a random variable, let g be a function defined on the range of X. We already know how to compute $E(g(X))$

$$E(g(X)) = \sum g(k)\,P(X=k) \qquad E(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$$

Q: How to compute the PMF/PDF of $g(X)$?

Discrete case: (i) find the set of all possible values of $g(X)$

(ii)

Example

k	-1	0	1	2
$P(X=k)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$g(x) = x^2$

ℓ	0	1	2
$P(X^2=\ell)$			

Distribution of a function of a continuous random variable

Let X be a continuous random variable, $g: \mathbb{R} \to \mathbb{R}$

In order to compute the PDF of $g(X)$

(i) compute the CDF of $g(X)$

(ii) differentiate the CDF of $g(X)$

Example

Let $U \sim \text{Unif}[0,1]$, let $g(x) = -\frac{1}{\lambda} \log(1-x)$ for $\lambda > 0$

Find the distribution of $Y = g(X)$.

(0) Range of (the possible values) of Y is

(i)

(ii)

Linear transformation of a continuous random variable

Example Let X be a continuous random variable with PDF $f_X(x)$ and CDF $F_X(x)$. Let $Y = aX + b$ with $b \in \mathbb{R}$, $a \neq 0$.

Compute CDF and PDF of Y. $Y = g(X)$ with $g(x) = ax + b$

$F_Y(y) =$

$f_Y(y) =$

General formula for a function of a continuous RV

Let X be a continuous random variable with PDF $f_X(x)$

If $g: \mathbb{R} \to \mathbb{R}$ is one-to-one, differentiable and $g'(x) = 0$ only on a finite set, then

$$f_{g(X)}(y) = f_X\left(g^{-1}(y)\right) \cdot \frac{1}{\left|g'\left(g^{-1}(y)\right)\right|}$$

<u>Example</u> Let $X \sim N(0,1)$, $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^3$. Find PDF of X^3.

Let $Y = g(X) = X^3$. Y takes values on the whole $\mathbb{R}$.

If g is not one-to-one, split into intervals where g is one-to-one.

Random vectors

Until now we studied (mostly) individual random variables, one at a time, using various tools such as

PMF/PDF, CDF, expectation, variance, moments, MGF

We discussed some very simple models with finite/infinite number of random variables (independent trials)

New setting: random variables $X_1, X_2, X_3, \dots, X_n$, all defined on the same probability space (not necessarily independent)

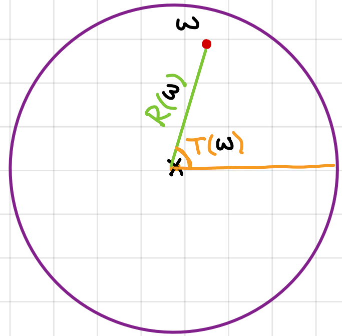
$(\Omega, \mathcal{F}, P)$, $\underline{X}: \Omega \to \mathbb{R}^n$,

Example

- Choose a point ω from a unit disk

$$R(\omega) = \text{distance to the center}$$

$$T(\omega) = \text{angle}$$



(R, T) is a random vector

- Roll a fair die 2 times

$$X_1 = \#\text{ of even numbers}$$

$$X_2 = \#\text{ of sixes}$$

(X_1, X_2) is a random vector

Joint distribution (discrete case)

Def. (Joint PMF). Let $X_1, X_2, \dots, X_n$ be discrete random variable defined on the same probability space.
The joint probability mass function of $(X_1, \dots, X_n)$ is given by

Remark

Example Roll a fair die twice

$X_1 = \#$ of even numbers

$X_2 = \#$ of sixes

$p_{X_1,X_2}(k_1,k_2)$

$k_1 \backslash k_2$	0	1	2
0			
1			
2			

Expectation of a function of a random vector

Let $X_1, \dots, X_n$ be discrete random variables. Let $g: \mathbb{R}^n \to \mathbb{R}$

Then $E(g(X_1, \dots, X_n)) =$

where we sum over all possible values $k_1, \dots, k_n$ of $X_1, \dots, X_n$

Example (cont.) For each even number you get 1 dollar, and the sum is multiplied by the number of sixes.

$g(k_1, k_2) =$

$E(g(X_1, X_2)) =$

$=$

Marginal PMF

Let $p(k_1, \dots, k_n)$ be a joint PMF of random variables $X_1, \dots, X_n$. Then for any $1 \le j \le n$ the marginal PMF of X_j is given by

(fix j-th variable, sum over all other variables)

Example

$p(k_1, k_2)$

k_1 \ k_2	0	1	2
0	$\frac{9}{36}$	0	0
1	$\frac{12}{36}$	$\frac{6}{36}$	0
2	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{36}$

Q:

Joint distribution of continuous random variables

Def. Random variables $X_1, \dots, X_n$ are jointly continuous if there exists a function $f: \mathbb{R}^n \to \mathbb{R}$ such that

Function $f(x_1, \dots, x_n)$ is called the joint density

Joint density satisfies :

Example Consider random variables X and Y with joint density

Expectation of a function. Marginal PDF

Let $X_1, \dots, X_n$ be jointly continuous random variables with joint PDF f_X. Let $g: \mathbb{R}^n \to \mathbb{R}$ be a function of n variables.
Then

<u>Def</u>. Let f be the joint density of $X_1, \dots, X_n$. Then each random variable X_j has a (marginal) density

(fix j-th variable, integrate all other variables)

Example

Consider again random variables X, Y with joint density

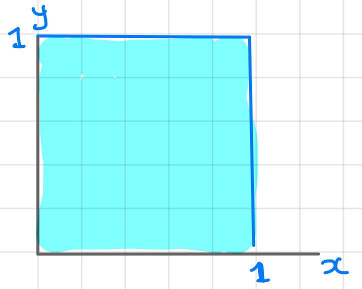
$$f(x,y) = \begin{cases} \frac{3}{2}(xy^2+y) & , \ 0 \le x \le 1, \ 0 \le y \le 1 \\ 0 & , \ \text{otherwise} \end{cases}$$

(i) $E(X^2Y) =$

$g(x,y) = x^2 y$

(ii) $f_X(x) =$

$f_Y(y) =$



(iii) $P(X < Y) =$