MATH 180A (Lecture A00)

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Today: Joint distribution. Independence

Next: ASV 8.1

Week 9:

- Homework 6 due Friday, March 10

Joint distribution (discrete case)

Def. (Joint PMF). Let $X_1, X_2, \dots, X_n$ be discrete random variable defined on the same probability space.

The joint probability mass function of $(X_1, \dots, X_n)$ is given by

$$p_{X_1, X_2, \dots, X_n}(k_1, k_2, \dots, k_n) = P(X_1 = k_1, X_2 = k_2, \dots, X_n = k_n)$$

for all possible values $(k_1, k_2, \dots, k_n)$ of $(X_1, X_2, \dots, X_n)$

Remark $\sum_{k_1, k_2 \dots k_n} p_{\underline{X}}(k_1, k_2, \dots, k_n) = 1$

Example Roll a fair die twice

$X_1 = \#$ of even numbers

$X_2 = \#$ of sixes

$p_{X_1, X_2}(k_1, k_2)$

$k_1 \backslash k_2$	0	1	2
0	$\frac{9}{36}$	0	0
1	$\frac{12}{36}$	$\frac{6}{36}$	0
2	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{36}$

Expectation of a function of a random vector

Let $X_1, \ldots, X_n$ be discrete random variables. Let $g: \mathbb{R}^n \to \mathbb{R}$

Then $$E(g(X_1, \ldots, X_n)) = \sum_{k_1, k_2, \ldots, k_n} g(k_1, k_2, \ldots, k_n)\, p_{\underline{X}}(k_1, k_2, \ldots, k_n)$$

where we sum over all possible values $k_1, \ldots, k_n$ of $X_1, \ldots, X_n$

Example (cont.) For each even number you get 1 dollar, and the sum is multiplied by the number of sixes.

$g(k_1, k_2) = k_1 \cdot k_2$ $\qquad E(X_1 X_2)$

$+ 0 \cdot 0 \cdot p(0,0) + 1 \cdot 0 \cdot p(1,0)$

$$E(g(X_1, X_2)) = 1 \cdot 1 \cdot p(1,1) + 2 \cdot 1 \cdot p(2,1) + 2 \cdot 2 \cdot p(2,2)$$

$$= 1 \cdot \frac{6}{36} + 2 \cdot \frac{4}{36} + 4 \cdot \frac{1}{36} = \frac{18}{36} = \frac{1}{2}$$

Marginal PMF

Let $p(k_1, \dots, k_n)$ be a joint PMF of random variables $X_1, \dots, X_n$. Then for any $1 \le j \le n$ the marginal PMF of X_j is given by

$$p_{X_j}(\ell) = \sum_{\substack{k_1, \dots, k_{j-1} \\ k_{j+1}, \dots, k_n}} p_{\underline{X}}(k_1, k_2, \dots, k_{j-1}, \overset{j\text{th}}{\ell}, k_{j+1}, \dots, k_n)$$

(fix j-th variable, sum over all other variables)

Example

$p(k_1, k_2)$

$k_1 \backslash k_2$	0	1	2	
0	$\frac{9}{36}$	0	0	$\frac{9}{36}$
1	$\frac{12}{36}$	$\frac{6}{36}$	0	$\frac{18}{36}$
2	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{36}$	$\frac{9}{36}$
	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$	

$p_{X_1}(0) = P(X_1 = 0) = p(0,0) + p(0,1) + p(0,2) = \frac{9}{36}$

$p_{X_1}(1) = p(1,0) + p(1,1) + p(1,2) = \frac{18}{36}$

$p_{X_1}(2) = p(2,0) + p(2,1) + p(2,2) = \frac{9}{36}$

Q: $P(X_1 X_2 \le 2) = 1 - P(X_1 X_2 > 2)$

$= 1 - P(X_1 X_2 = 4) = \frac{35}{36}$

$p_{X_2}(0) = p(0,0) + p(1,0) + p(2,0)$

Joint distribution of continuous random variables

Def. Random variables $X_1, \dots, X_n$ are **jointly continuous** if there exists a function $f: \mathbb{R}^n \to \mathbb{R}$ such that for any $B \subset \mathbb{R}^n$

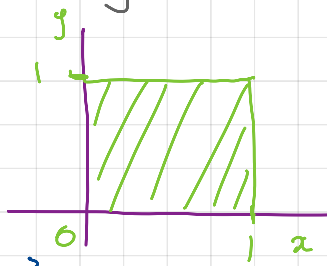
$$P\big((X_1, \dots, X_n) \in B\big) = \int \cdots \int_B f(x_1, \dots, x_n)\, dx_1 \cdots dx_n$$

Function $f(x_1, \dots, x_n)$ is called the **joint density**

Joint density satisfies: $f \geq 0$, $\displaystyle\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, \dots, x_n)\, dx_1 \cdots dx_n = 1$

Example Consider random variables X and Y with joint density

$$f(x,y) = \begin{cases} \frac{3}{2}(xy^2 + y), & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{3}{2}(xy^2+y)\, dx\, dy = \int_0^1 \left(\int_0^1 \frac{3}{2}(xy^2+y)\, dx \right) dy = \int_0^1 \left(\frac{3}{4} y^2 + y\frac{3}{2} \right) dy = \frac{1}{4} + \frac{3}{4} = 1$$

Expectation of a function. Marginal PDF

Let $X_1, \dots, X_n$ be jointly continuous random variables with joint PDF f_X. Let $g: \mathbb{R}^n \to \mathbb{R}$ be a function of n variables.

Then

$$E(g(X_1, \dots, X_n)) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g(x_1, \dots, x_n) f(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

<u>Def</u>. Let f be the joint density of $X_1, \dots, X_n$. Then each random variable X_j has a (marginal) density

$$f_{X_j}(x_j) = \underbrace{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}}_{n-1 \text{ times}} f_X(x_1, \dots, x_n)\, dx_1 \cdots dx_{j-1}\, dx_{j+1} \dots dx_n$$

(fix j-th variable, integrate all other variables)

Example

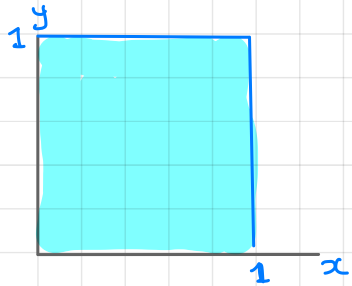
Consider again random variables X, Y with joint density

$$f(x,y) = \begin{cases} \frac{3}{2}(xy^2+y) & , \quad 0 \le x \le 1, \ 0 \le y \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

(i) $E(X^2 Y) = \int_0^1 \int_0^1 x^2 y \cdot \frac{3}{2}(xy^2+y)\, dx\, dy = \frac{25}{36}$

$g(x,y) = x^2 y$

(ii) $f_X(x) = \int_0^1 \frac{3}{2}(xy^2+y)\, dy = \frac{x}{2} + \frac{3}{4} \quad , \quad x \in (0,1)$



$f_Y(y) =$

(iii) $P(X < Y) =$