MATH 180A (Lecture A00)

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Today: Joint distribution. Independence

Next: ASV 8.1

Week 9:

- Homework 6 due Friday, March 10

Example

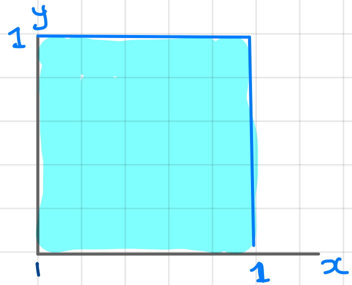
Consider again random variables X, Y with joint density

$$f(x,y)=\begin{cases} \frac{3}{2}(xy^2+y) & , \ 0\le x\le 1,\ 0\le y\le 1 \\ 0 & , \ \text{otherwise}\end{cases}$$

(i) $E(X^2Y) = \int_0^1\int_0^1 x^2y\,\frac{3}{2}(xy^2+y)\,dx\,dy = \frac{3}{2}\int_0^1\int_0^1(x^3y^3+x^2y^2)\,dx\,dy = \cdots = \frac{25}{36}$

$g(x,y)=x^2y$

(ii) $f_X(x) = \int_0^1 \frac{3}{2}(xy^2+y)\,dy = \frac{3}{2}\left(x\cdot\frac{1}{3}+\frac{1}{2}\right) = \frac{x}{2}+\frac{3}{4}, \quad x\in[0,1]$



$f_Y(y) = \int_0^1 \frac{3}{2}(xy^2+y)\,dx = \frac{3}{4}y^2+\frac{3}{2}y, \quad y\in[0,1]$

(iii) $P(X<Y) =$

Joint distribution and independence

Random variables $X_1, \dots, X_n$ defined on the same probability space are independent if for any $B_1, \dots, B_n \subset \mathbb{R}$

$$P(X_1 \in B_1, X_2 \in B_2, \cdots, X_n \in B_n) = P(X_1 \in B_1) P(X_2 \in B_2) \cdots P(X_n \in B_n)$$

Independence can be expressed in terms of PMF/PDF

Discrete case:

Let $p(k_1, \dots, k_n)$ be the joint PMF of discrete random variables $X_1, \dots, X_n$. Let $p_{X_j}(k) = P(X_j = k)$ be the marginal PMF of X_j. Then

if and only if

Examples

Example 1

Roll a fair die twice

X_1 = # even numbers

X_2 = # sixes

$k_1 \backslash k_2$	0	1	2	
0	$\frac{9}{36}$	0	0	$\frac{9}{36}$
1	$\frac{12}{36}$	$\frac{6}{36}$	0	$\frac{18}{36}$
2	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{36}$	$\frac{9}{36}$
	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$	

Are X_1 and X_2 independent?

Example 2

joint PMF of X_1, X_2

$k_1 \backslash k_2$	0	1	
0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
1	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$
	$\frac{1}{3}$	$\frac{2}{3}$	

for any k_1, k_2

X_1 and X_2

Joint distributions and independence. Continuous case

Thm. Let $X_1, \dots, X_n$ be random variables defined on the same probability space. Assume that each X_j has PDF f_{X_j}.

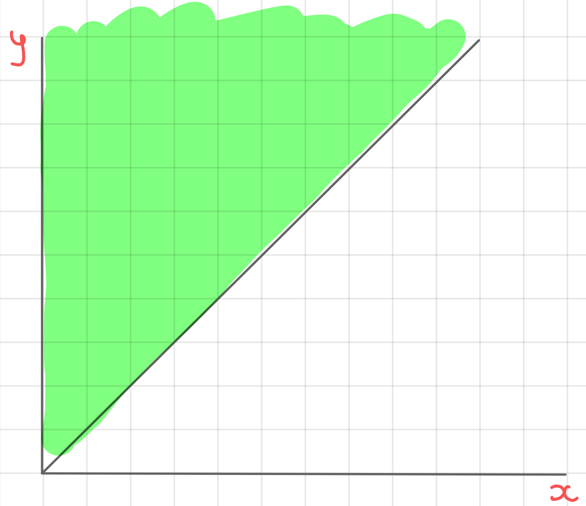
(i) If the joint density of $X_1, \dots, X_n$ is equal to

then $X_1, \dots, X_n$ are independent

(ii) If $X_1, \dots, X_n$ are independent then

Thm. Let $X_1, \dots, X_n, X_{n+1}, \dots, X_{n+m}$ be independent random variables. Let $f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^m \to \mathbb{R}$. Define $Y = f(X_1, \dots, X_n)$, $Z = g(X_{n+1}, \dots, X_{n+m})$. Then

Joint distribution and independence



Example Let $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$

Suppose X and Y are independent.

1) Find the joint PDF of X and Y
2) Compute $P(X<Y)$

X takes values in $(0,+\infty)$, Y takes values in $(0,+\infty)$

(X,Y) take values in $\{(x,y): x>0, y>0\}$

1) From the independence $f_{X,Y}(x,y) =$

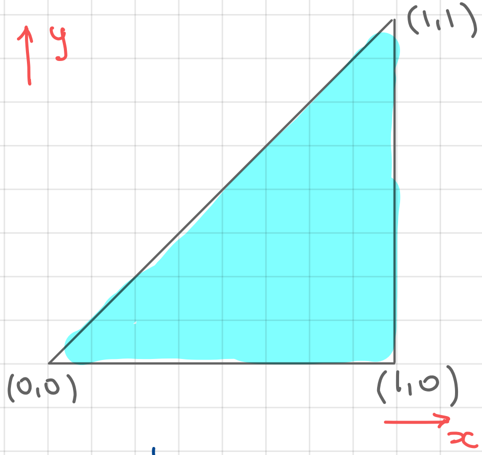
2) $P(X<Y) = P((X,Y) \in \{(x,y): x<y\}) =$

$$= \int_0^\infty \lambda e^{-\lambda x} \int_x^\infty \mu e^{-\mu y} dy \, dx = \int_0^\infty \lambda e^{-\lambda x} e^{-\mu x} dx = \int_0^\infty \lambda e^{-(\lambda+\mu)x} dx$$

Joint distribution of $X_1, \dots, X_n$. Summary

Discrete	Continuous
Joint distribution	
Joint PMF	**Joint PDF**
$p(k_1, k_2, \dots, k_n)$ $= P(X_1 = k_1, X_2 = k_2, \cdots, X_n = k_n)$	$P((X_1, X_2, \dots, X_n) \in B)$ $= \int \cdots \int_B f(x_1, \dots, x_n)\, dx_1 \cdots dx_n$
Marginal distribution	
Marginal PMF	**Marginal PDF**
$p_{X_j}(\ell) = \sum_{k_1} \cdots \sum_{k_{j-1}} \sum_{k_{j+1}} \cdots \sum_{k_n} p(k_1, \cdots, k_{j-1}, \ell, k_{j+1}, \dots, k_n)$	$f_{X_j}(y) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n)\, dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n$
Independence	
$p(k_1, \dots, k_n) = p_{X_1}(k_1) \cdots p_{X_n}(k_n)$	$f(x_1, \dots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$

Joint distribution and independence



Exercise Let X and Y be jointly continuous random variables with joint PDF

$$f(x,y) = \begin{cases} c\,xy, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

1) Find constant c

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = c\int_0^1\int_0^x xy\,dy\,dx$$

2) Find marginal densities of X and Y

3) Are X and Y independent?

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy =$$

$$f_Y(y) =$$

Joint distribution and independence

Exercise Let the joint distribution of random variables X and Y be given by the joint PMF $p(k,\ell) = P(X=k, Y=\ell)$

$k \backslash \ell$	0	1	2
−1	0.1	0.1	0.2
1	0.2	0.1	α

$$\sum_{k,\ell} p(k,\ell) = 1$$

1) Find unknown α

2) Compute the marginal PMFs of X and Y

k	−1	1
$P(X=k)$		

ℓ	0	1	2
$P(Y=\ell)$			

3) Are X and Y independent?

Joint distribution and independence

Example Let $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$

Suppose X and Y are independent.

Find the distribution of $Z = \min\{X, Y\}$

$$P(Z > t) = P(\min\{X, Y\} > t) =$$

Linearity of expectation

<u>Thm</u>. Let $X_1, \dots, X_n$ be random variables defined on the same probability space. Let $g_1, g_2, \dots, g_n$ be functions of one variable.

Then

$$E(g_1(X_1) + g_2(X_2) + \cdots + g_n(X_n))$$

In particular, $E(X_1 + \cdots + X_n)$

Important: independence does not matter

Expectation of a sum = sum of expectations

ALWAYS!

Linearity of expectation

Example (Binomial distribution).

Let $X_1, \dots, X_n$ be independent random variables, $X_i \sim Ber(p)$

$$S_n = X_1 + \cdots + X_n, \quad S_n \sim Bin(n,p).$$

$$E(S_n) =$$

Example Adam must pass both written test and road test for his driver's license. He passes written test with probability $\frac{4}{10}$, independently of other tests. For the road test, the probability of success is $\frac{7}{10}$.

What is the total expected number of tests Adam must take before earning his license?

Denote $X = \#$ written tests before he passes

$Y = \#$ road tests before he passes

$$E(X+Y) - ?$$

Expectation of a product of independent random variables

<u>Thm</u>. Let $X_1, \ldots, X_n$ be independent random variables.

Let $g_1, g_2, \ldots, g_n$ be functions of one variable.

Then

$$E\left(g_1(X_1)\, g_2(X_2) \cdots g_n(X_n)\right)$$

<u>Corollary</u> If $X_1, \ldots, X_n$ are independent, then

Variance of a sum of independent random variables

Example

Binomial: $X_1, \dots, X_n$ independent identically distributed (iid)

$X_i \sim Ber(p)$, $Var(X_i) = p(1-p)$, $S_n = X_1 + \dots + X_n$

$Var(S_n) =$

Sample mean: $X_1, \dots, X_n$ independent identically distributed (iid)

$E(X_i) = \mu$, $Var(X_i) = \sigma^2$

$E\left(\frac{X_1 + \dots + X_n}{n}\right) =$, $Var\left(\frac{X_1 + \dots + X_n}{n}\right) =$

Covariance

Suppose that we have a random variable X.

- $E(X)$ – mean value, average of a large number of independent realizations
- $Var(X)$ – variance, fluctuations of X, how far the realizations are spread around the mean

Covariance describes strength and type of dependence between two random variables.

Def. Let X and Y be random variables defined on the same probability space. The covariance of X and Y is given by

Computations:

Covariance

Example Let X, Y be discrete random variables with the joint PMF $P(X=k, Y=\ell)$ given by the table

$k \backslash \ell$	0	1	2	
-1	0.1	0	0.1	0.2
0	0.3	0.2	0	0.5
1	0	0.2	0.1	0.3
	0.4	0.4	0.2	

$E(X) =$

$E(Y) =$

$E(XY) =$

$\mathrm{Cov}(X, Y)$