MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Definition of probability. Random sampling

Next: ASV 1.3-1.4

Week 1:

- check the course website
- homework 1 (due January 20)
- join Piazza

Axioms of probability

Mathematical model of an experiment with random outcome

<u>Def.</u> Probability space is the triple $(\Omega, \mathcal{F}, P)$, where

- Ω is the set of all possible outcomes of the experiment; we call it the sample space
- $\mathcal{F}$ is a collection of subsets of Ω (events)
- P is a function that assigns to each event a real number and satisfies the following properties:

AXIOMS OF PROB

(i) $0 \le P(A) \le 1$ for all $A \in \mathcal{F}$

(ii) $P(\emptyset) = 0$, $P(\Omega) = 1$

(iii) If $A_1, A_2, A_3, \dots$ are disjoint events, then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

P is the probability measure (or simply probability)

Examples

Example 2: rolling a fair die

$\Omega = \{1,2,3,4,5,6\}$, $\mathcal{F} = \{\text{all subsets of } \Omega\}$

$$P(\{1\}) = P(\{2\}) = \dots = P(\{6\}) = \frac{1}{6}$$

What about the events? Take $A = \{2,4,6\} \subset \Omega$.

$$P(A) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

$B = \{2,3,5\}$: $P(B) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

$C = \{3,6\}$: $P(C) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

A = "even number"

B = "prime number"

C = "divisible by 3"

$$P(A \cup B) = P(\{2,3,4,5,6\}) = \frac{5}{6}$$

$$P(B \cap C) = P(\{3\}) = \frac{1}{6}$$

Repeated experiments

What is the sample space if we toss the coin twice?

The outcome is a pair with

The collection of such pairs is called the Cartesian product of $\{H,T\}$ and $\{H,T\}$, denoted $\{H,T\}\times\{H,T\}$

$$\{H,T\}\times\{H,T\} = \{(H,H),(H,T),(T,H),(T,T)\}$$

sample space

More generally, for any sets $\Omega_1, \Omega_2, \dots, \Omega_k$

sample space if we perform experiment 1 with s.s. Ω_1

experiment 2 with s.s. Ω_2

$\vdots$

experiment k with s.s. Ω_k

Finite sample space

Consider a special case when $\#\Omega < \infty$. Then

$\Omega =$ for $n = \#\Omega$

Any event $A \subset \Omega$ is a finite union of $\{\omega_i\}$.
The singleton sets $\{\omega_1\}, \dots, \{\omega_n\}$ are disjoint.

Therefore, if $A = \{a_1, \dots, a_k\}$ for some $a_i \in \Omega$, then

$P(A) =$

What if additionally we have that $P(\{\omega_1\}) = \dots = P(\{\omega_n\})$?

Uniform probability measure and random sampling

If Ω is finite, the uniform probability measure is defined by the following property:

$$\text{for each } \omega \in \Omega, \quad P(\{\omega\}) =$$

From (*) this implies that

$$\text{for any event } A, \quad P(A) =$$

This means that for such models calculating probabilities is reduced to counting.

<u>Example</u> Roll a fair dice twice. What is the probability that the sum is 4? $\Omega = \{(i,j) : 1 \leq i,j \leq 6\}$,

$A = \{(1,3),(2,2),(3,1)\}$, $\qquad\qquad P(A) =$

Uniform probability measure and random sampling

Example A fair coin is tossed 3 times.

$A = \{\text{at least two tails}\}$

$B = \{\text{exactly two tails}\}$

$\Omega = \{(x_1, x_2, x_3) : x_i \in \{H, T\}\}$, $\#\Omega =$

$A =$, $\#A =$

$B =$, $\#B =$

$P(A) =$, $P(B) =$

Warm-up exercise

There are 10 people on a committee.

How many different ways are there to select a subcommittee of 4 people?

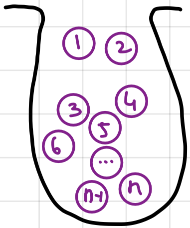
(a) $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10\,000$

(b) $10 \cdot 9 \cdot 8 \cdot 7 = 5040$

(c) $\binom{10}{4} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = 210$

(d) $\dfrac{10!}{4!} = 151\,200$

Combinatorics



A collection of n labelled balls $\{1, 2, 3, \ldots, n\}$ are in an urn. k are taken out one by one.

Q: How many ways?

Possible scenarios:

Replacement

- with replacement
- without replacement

Order

- order balls come out matters
- order does not matter

$n = 5$, $k = 3$ (choose 3 balls)	order matters	order doesn't matter
with replacement	○○○ ○○○	○○○ ○○○
without replacement	○○○ ○○○	○○○ ○○○

Combinatorics

Sampling with replacement, order matters

sample space : $\Omega =$

Sampling without replacement, order matters

sample space : $\Omega =$

Sampling without replacement, order does not matter

sample space : $\Omega =$

	order matters	order doesn't matter
with replacement	$\#\Omega =$	
without replacement	$\#\Omega =$ $=$	$\#\Omega =$ $=$

Important remark. Examples

Each of these three models leads to a

uniform probability measure!

on the corresponding sample space

<u>Example</u> (sampling with replacement)

Toss a fair coin n times; record a statistic observing

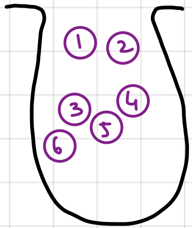
$\# H$ vs $\# T$

Take $n=10$. Q: compute P(odd rolls are all H)

$\Omega = \{ (c_1, c_2, \dots, c_{10}) : c_j \in \{H, T\} \}$, $\#\Omega =$

Examples

Example (Sampling without replacement, order matters)



There are 6 labelled balls in an urn.

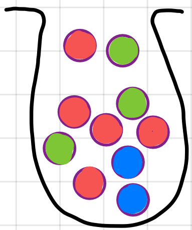
3 are removed in sequence (without replacement) and lined up in order.

Q: What is the probability that the first two are (4, 5)?

$$\Omega = \{ (b_1, b_2, b_3) : 1 \le b_j \le 6,\ b_1 \neq b_2,\ b_2 \neq b_3,\ b_1 \neq b_3 \}$$

Examples

Example (Sampling without replacement, order does not matter)



An urn contains 10 balls: $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}$

two blue, three green, five red

3 balls are chosen without replacement.

Q: Compute P(choose 2 green and one red)

$\# \Omega =$