MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

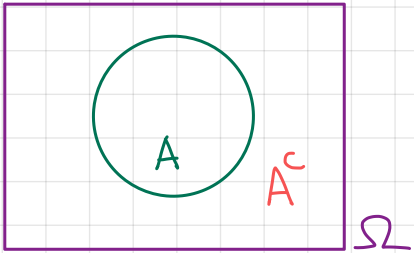
Today: Conditional probability

Next: ASV 2.2

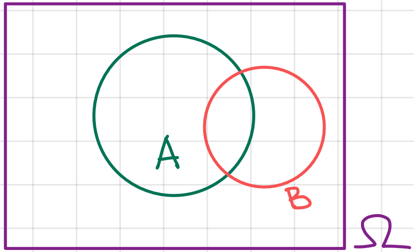
Week 2:

- homework 1 (due Monday, January 23)

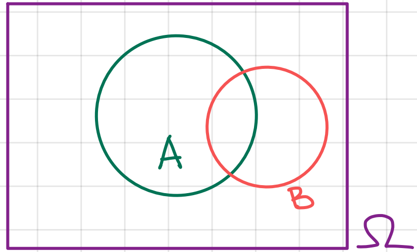
Intersections and unions



$A \cup B$



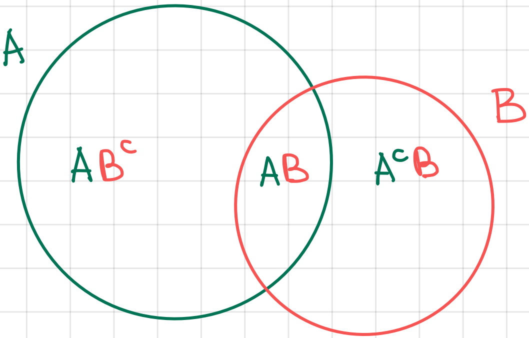
$A \cap B$



Sometimes we have to take intersections into account.

Notation: $A \cup B = \{\text{all outcomes in either } A \text{ or } B \text{ or both}\}$

$A \cap B = \{\text{all outcomes in both } A \text{ and } B\} = AB$



$$A \cup B = AB^c \cup AB \cup A^cB$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Principle of inclusion / exclusion

Example Among students enrolled in MATH 180A

60% are Math majors

20% are Physics majors

5% are majoring in both Math and Physics

A student is chosen randomly from the class. What is the probability that this student is neither a Math major nor a Physics major?

$A = \{\text{Math}\}$

$B = \{\text{Physics}\}$

$C = \{\text{neither}\}$

$C = (A \cup B)^c \Rightarrow P(C) = 1 - P(A \cup B)$

$P(A) = 0.6$

$P(B) = 0.2$

$P(A \cap B) = 0.05$

$\Rightarrow P(A \cup B) = 0.6 + 0.2 - 0.05 = 0.75$

$P(C) = 1 - 0.75 = 0.25$

Warm-up question

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

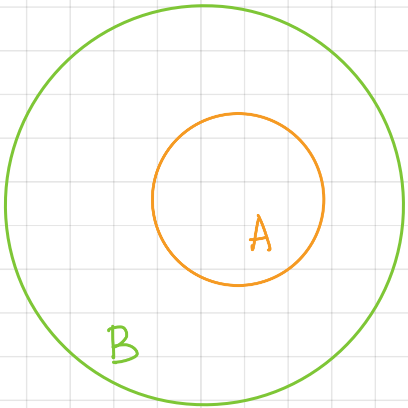
$A = \{\text{bank teller}\}$

$B = \{\text{active in the feminist movement}\}$

Which is more probable?

1. Linda is a bank teller. A
2. Linda is a bank teller and is active in the feminist movement. $A \cap B$

Monotonicity



$B = A \cup (B \setminus A)$

If $A \subseteq B$, then $P(A) \leq P(B)$

Indeed, $P(B) = P(A) + P(B \setminus A)$
$\geq P(A)$

For any A, B, $P(A \cup B) \geq \max\{P(A), P(B)\}$
$P(A \cap B) \leq \min\{P(A), P(B)\}$

Useful tools

- $P(A) + P(A^c) = 1$ (events and their complements)
- $A \subset B$ implies $P(A) \leq P(B)$ (monotonicity)
- $P(A \cup B) = P(A) + P(B) - P(AB)$ (inclusion-exclusion)

Conditional probability

Example 1. Your friend rolls two fair dice. What is the probability the sum is 10.

$$\Omega = \{(i,j) : 1 \le i,j \le 6\} \qquad A = \{(i,j) : 1 \le i,j \le 6,\ i+j=10\}$$

$$P(A) = \frac{3}{36} = \frac{1}{12} \qquad = \{(5,5),(4,6),(6,4)\}$$

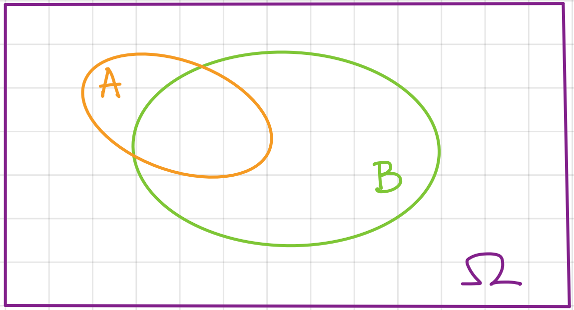
Example 2 Your friend rolls two fair dice and tells you that the sum that came up is a two digit number. What is the probability that the sum is 10?

"Updated" $\tilde{\Omega} = \{\text{sum has 2 digits}\} = \{(4,6),(5,5),(6,4),(5,6),(6,5),(6,6)\}$

$$\tilde{P}(A) = \frac{\# A}{\# \tilde{\Omega}} = \frac{3}{6} = \frac{1}{2}$$

(sometimes you need to update A)

Conditional probability



If we know that the event B happened

- keep the same Ω and $\mathcal{F}$
- define new probability $\tilde{P}$ on $(\Omega, \mathcal{F})$ that takes into account the additional information
  - probability of "everything" that is not part of B is set to zero
  - if $C \subset B$, $\tilde{P}(C) = \dfrac{P(C)}{P(B)}$

<u>Def</u>. Let $B \in \mathcal{F}$ satisfy $P(B) > 0$. Then for all $A \in \mathcal{F}$ the conditional probability of A given B is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Conditional probability

If $P(B)>0$, then the conditional probability $P(\cdot\,|\,B)$ satisfies all the properties of probability: axioms of probability, probability of the complement, monotonicity, inclusion-exclusion ...

Remark If Ω is finite and P is uniform, then

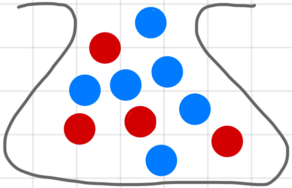
$$\frac{\frac{\#(A\cap B)}{\cancel{\#\Omega}}}{\frac{\#B}{\cancel{\#\Omega}}} = \frac{P(A\cap B)}{P(B)} = P(A\,|\,B) = \frac{\#(A\cap B)}{\#B}$$

Example Roll two fair dice. $A=\{\text{sum is } 10\}$,

$B=\{\text{sum is a two-digit number}\}$

$\#A=3$, $\#B=6$, $\#A\cap B=3$, $P(A\,|\,B)=\frac{3}{6}=\frac{1}{2}$

Examples



An urn contains 4 red balls and 6 blue balls. Three balls are sampled without replacement. What is the probability that exactly two are red?

$A = \{\text{exactly 2 are red}\}$

$\#\Omega = \binom{10}{3} = 120 \qquad \#A = \binom{4}{2}\binom{6}{1} = 36, \quad P(A) = \frac{36}{120} = \frac{3}{10}$

Suppose we know a priori that at least one red ball is sampled. What is the conditional probability of A?

$B = \{\text{at least one red ball}\}$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{3}{10}}{\frac{5}{6}} = \frac{18}{50} = 0.36$$

$$P(B) = 1 - P(B^c) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(B^c) = \frac{\binom{6}{3}\binom{4}{0}}{\binom{10}{3}} = \frac{20}{120} = \frac{1}{6}$$

$A \subset B \Rightarrow A \cap B = A, \quad P(A \cap B) = P(A) = 0.3$

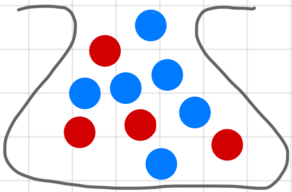
Multiplication rule

By definition $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

$$\Rightarrow \quad P(A \cap B) = P(A) \cdot P(B|A) \leftarrow$$ multiplication rule

For $A \cap B \cap C$: $P(A \cap B \cap C) = P(AB)\,P(C|AB) = P(A)\,P(B|A)\,P(C|AB)$

Example

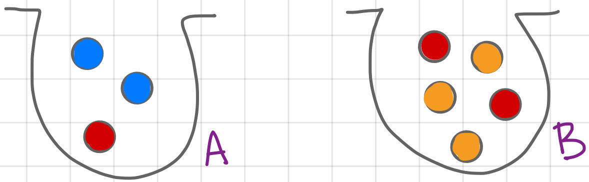


An urn contains 4 red ball and 6 blue balls. Two balls are sampled without replacement. What is the probability that both are red?

Two-stage experiments

- perform an experiment, measure a random outcome
- perform a second experiment whose setup depends on the outcome of the first!

Example



- first choose an urn at random
- then sample a ball at random from the chosen urn

Q: What is the probability that the sampled ball is red?

$R = \{\text{sample red ball}\}$, $A = \{\text{choose urn } A\}$, $B = \{\text{choose urn } B\}$