MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

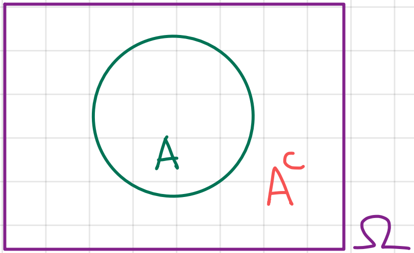
Today: Conditional probability

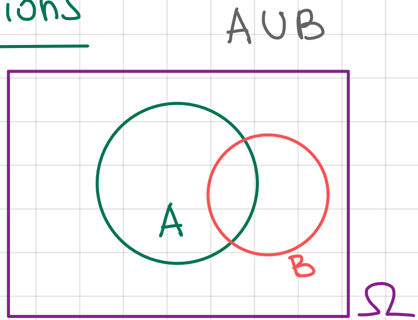
Next: ASV 2.2

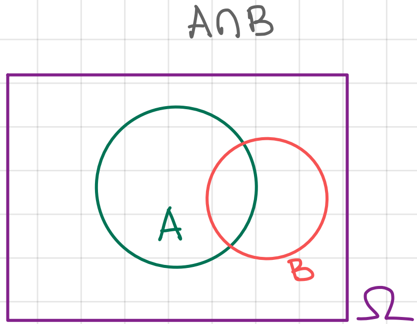
Week 2:

- homework 1 (due Monday, January 23)

Intersections and unions



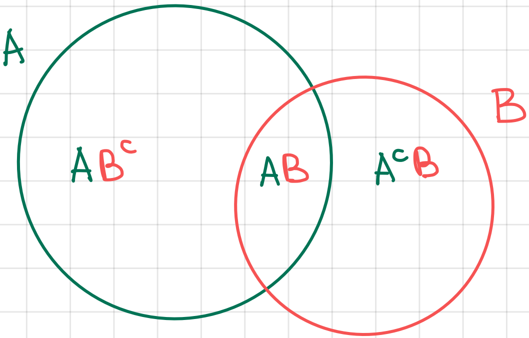




Sometimes we have to take intersections into account.

Notation: $A \cup B = \{\text{all outcomes in either } A \text{ or } B \text{ or both}\}$

$A \cap B = \{\text{all outcomes in both } A \text{ and } B\} = AB$



$$A \cup B = AB^c \cup AB \cup A^cB$$

Principle of inclusion / exclusion

Example Among students enrolled in MATH 180A

60 % are Math majors

20 % are Physics majors

5 % are majoring in both Math and Physics

A student is chosen randomly from the class. What is the probability that this student is neither a Math major nor a Physics major?

$A = \{ \text{Math} \}$

$B = \{ \text{Physics} \}$

$C = \{ \text{neither} \}$

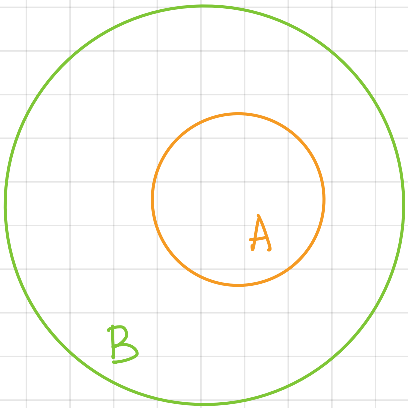
Warm-up question

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

Which is more probable?

1. Linda is a bank teller.
2. Linda is a bank teller and is active in the feminist movement.

Monotonicity



If $A \subseteq B$, then

Indeed, $P(B) =$

$\geq$

In particular, $P(A \cup B) \geq \max\{P(A), P(B)\}$

$P(A \cap B) \leq \min\{P(A), P(B)\}$

Useful tools

- $P(A) + P(A^c) = 1$ (events and their complements)
- $A \subset B$ implies $P(A) \leq P(B)$ (monotonicity)
- $P(A \cup B) = P(A) + P(B) - P(AB)$ (inclusion - exclusion)

Conditional probability

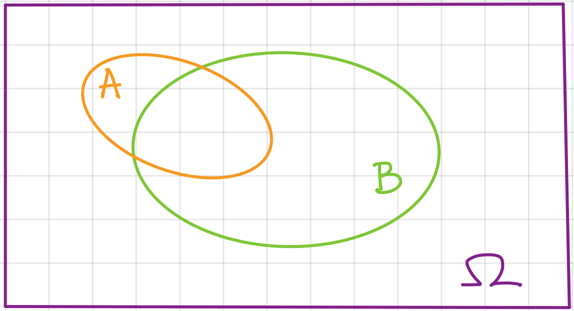
Example 1. Your friend rolls two fair dice. What is the probability the sum is 10.

$\Omega =$ $\qquad\qquad$ $A =$

$P(A) =$

Example 2 Your friend rolls two fair dice and tells you that the sum that came up is a two digit number. What is the probability that the sum is 10?

Conditional probability



If we know that the event B happened

- keep the same Ω and $\mathcal{F}$
- define new probability $\tilde{P}$ on $(\Omega, \mathcal{F})$ that takes into account the additional information

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<u>Def</u>. Let $B \in \mathcal{F}$ satisfy $P(B) > 0$. Then for all $A \in \mathcal{F}$ the conditional probability of A given B is defined as

Conditional probability

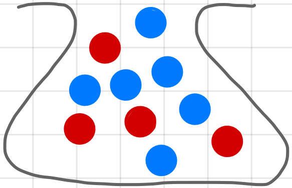
If $P(B) > 0$, then the conditional probability $P(\cdot \mid B)$ satisfies all the properties of probability:
axioms of probability, probability of the complement, monotonicity, inclusion-exclusion ...

Remark If Ω is finite and P is uniform, then

$$P(A \mid B) =$$

Example Roll two fair dice. $A = \{\text{sum is } 10\}$,
$B = \{\text{sum is a two-digit number}\}$

Examples



An urn contains 4 red ball and 6 blue balls. Three balls are sampled without replacement. What is the probability that exactly two are red?

Suppose we know a priori that at least one red ball is sampled. What is the conditional probability of A?

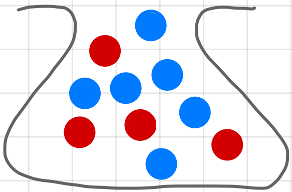
Multiplication rule

By definition $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

$\Rightarrow$

For $A \cap B \cap C$: $P(A \cap B \cap C) =$

Example

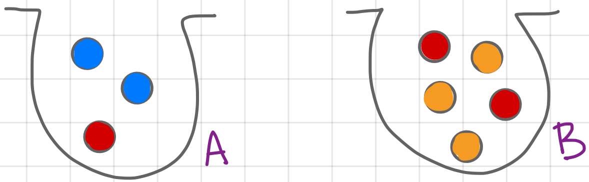


An urn contains 4 red ball and 6 blue balls. Two balls are sampled without replacement. What is the probability that both are red?

Two-stage experiments

- perform an experiment, measure a random outcome
- perform a second experiment whose setup depends on the outcome of the first!

Example



- first choose an urn at random
- then sample a ball at random from the chosen urn

Q: What is the probability that the sampled ball is red?

$R = \{\text{sample red ball}\}$, $A = \{\text{choose urn } A\}$, $B = \{\text{choose urn } B\}$

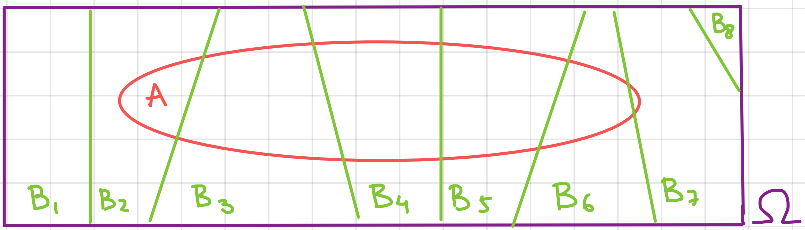
Law of total probability

Let $B_1, B_2, \dots, B_n$ be a partition of Ω

(i.e., B_i are disjoint, $B_1 \cup B_2 \cup \dots \cup B_n = \Omega$, $P(B_i) > 0$).

Then for every event A:

$P(A) =$



Example 90% of coins are fair, 9% are biased to come up heads 60% of times, 1% are biased to come up heads 80%. You find a coin on the street.

How likely is it to come up heads?

Law of total probability

Define $A = \{\text{coin comes up heads}\}$, $B_1 = \{\text{coin is fair}\}$

$B_2 = \{\text{coin is 60\% biased}\}$ $B_3 = \{\text{coin is 80\% biased}\}$

- B_1, B_2, B_3 form a , and

- $P(A \mid B_1) =$, $P(A \mid B_2) =$, $P(A \mid B_3) =$

Then using the law of total probability

$$P(A) =$$

$$=$$

Another question: In the same setting, you find a coin and toss it. It comes up heads. How likely is it that this coin is 80% biased (heavily biased)?

Important remark

We know that $P(A|B_3) = 0.8$.

What can we say about $P(B_3|A)$?

Generally speaking,

Example According to Forbes, there are 2668 billionaires in the world, 2357 of them are men.

Example Prosecutor's fallacy:

$E = \{\text{evidence on the defendant}\}$

$I = \{\text{defendant is innocent}\}$