MATH 180A (Lecture A00)

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Today: Random variables

Next: ASV 3.2

Week 3:

- homework 2 (due Sunday, January 29)
- Midterm 1 (Wednesday, February 1, lectures 1-8)
- 5 homework extension days per student per quarter

Independence for more than two events

<u>Def</u>. A collection of events $A_1, A_2, \ldots, A_n$ is **mutually independent** if for any subcollection of events $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ with $1 \le i_1 < i_2 < \cdots < i_k \le n$

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k})$$

<u>Example</u> For $n=3$, A, B, C are mutually independent if

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cap C) = P(A)P(C)$$

$$P(B \cap C) = P(B)P(C)$$

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Suppose that A and B are independent, A and C are independent, B and C are independent. Are A, B, C mutually independent?

Important example

Toss a coin

$A = \{$there is exactly one tails in the first two tosses$\}$

$B = \{$there is exactly one tails in the last two tosses$\}$

$C = \{$there is exactly one tails in the first and last tosses$\}$

$A = \{(H,T,*),(T,H,*)\}$ $\quad B = \{(*,H,T),(*,T,H)\}$

$C = \{(H,*,T),(T,*,H)\}$

$$P(A) = \frac{4}{8} = \frac{1}{2} = P(B) = P(C)$$

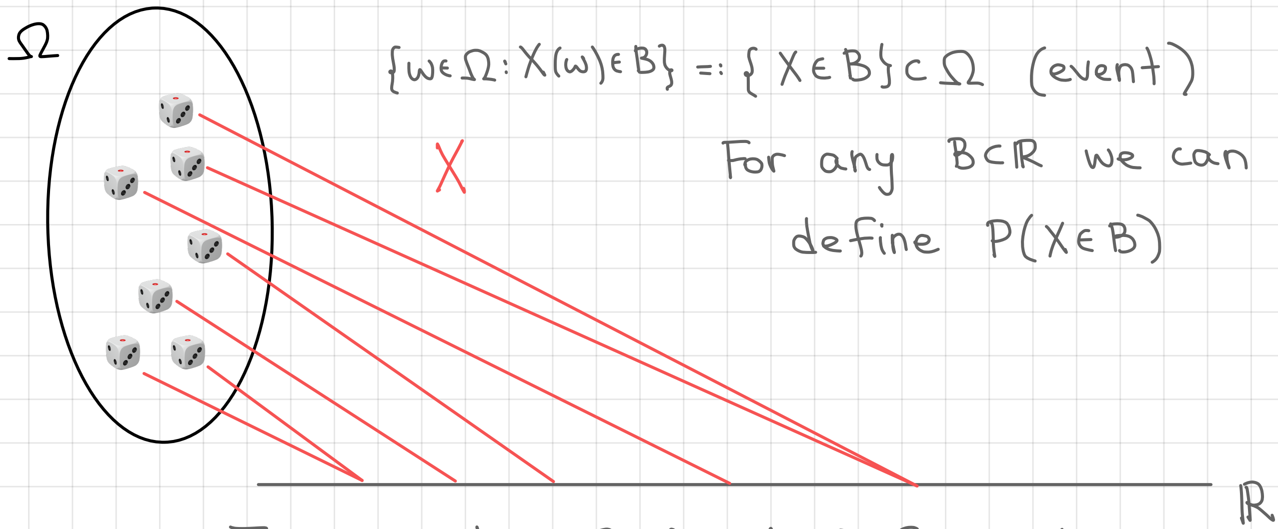
$$P(A \cap B) = \frac{2}{8} = \frac{1}{4} = P(B \cap C) = P(A \cap C),$$

$$P(A \cap B \cap C) = 0$$

Random variables

$(\Omega, \mathcal{F}, P)$ - probability space

<u>Def</u> A (measurable^{+}) function $X : \Omega \to \mathbb{R}$ is called a random variable.



$\{\omega \in \Omega : X(\omega) \in B\} =: \{X \in B\} \subset \Omega$ (event)

For any $B \subset \mathbb{R}$ we can define $P(X \in B)$

<u>Example</u> Toss a coin. $\Omega = \{H, T\}$. Define $X : \Omega \to \mathbb{R}$

Probability distribution

Def Let X be a random variable. The probability distribution of X is the collection of probabilities

Remark

Examples 1) Coin toss : $\Omega = \{H, T\}$, $X(H) = 1$, $X(T) = 0$

(fair coin)

2) Roll a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$,

For any $1 \leq i \leq 6$,

Probability distribution

3) Roll a die twice : $\Omega = \{(i,j) : i,j \in \{1,2,\ldots,6\}\}$

Define

$P(S=2) = \frac{1}{36}$ $\quad$ $P(S=7) = \frac{6}{36}$

$P(S=3) = \frac{2}{36}$ $\quad$ $P(S=8) = \frac{5}{36}$

$P(S=4) = \frac{3}{36}$ $\quad$ $P(S=9) = \frac{4}{36}$

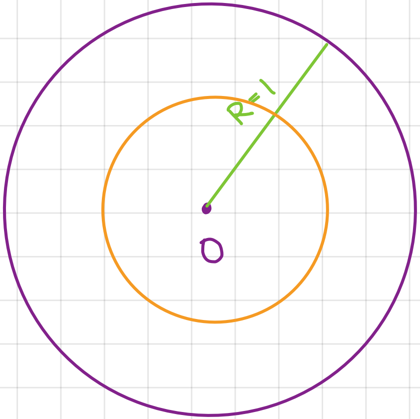
$P(S=5) = \frac{4}{36}$ $\quad$ $P(S=10) = \frac{3}{36}$

$P(S=6) = \frac{5}{36}$ $\quad$ $P(S=11) = \frac{2}{36}$

$P(S=12) = \frac{1}{36}$

Probability distribution

4) Choosing a point from unit disk uniformly at random



$\Omega = \{\omega \in \mathbb{R}^2 : \text{dist}(0, \omega) \leq 1\}$

For any $r < 0$,

For any $r > 1$,

For any $0 \leq r \leq 1$, $P(X \leq r) =$

Probability distribution

If $(\Omega, \mathcal{F}, P)$ is a probability space, and $X : \Omega \to \mathbb{R}$ is a random variable, we can define a probability measure μ_X on $\mathbb{R}$ given, for any $A \subset \mathbb{R}$, by

We call μ_X the probability distribution (or law) of X.

5) Toss a fair coin 4 times.

$\Omega = \{(X_1, X_2, X_3, X_4) \in \{H, T\}^4\}$

P = uniform on Ω

$P((X_1, X_2, X_3, X_4)) =$

If $A \subset \mathbb{R}$ does not contain one of these numbers, then

Enough to know

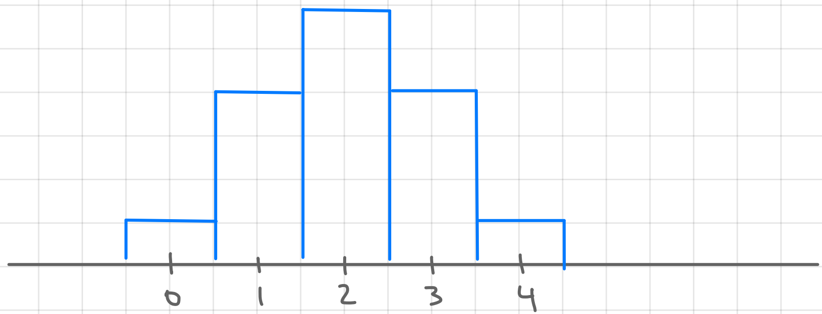
Probability distribution

Toss a fair coin 4 times. Let X = number of tails.

$X \in \{0, 1, 2, 3, 4\}$

$p_X(k) = P(X=k)$

k	0	1	2	3	4
$p_X(k)$					



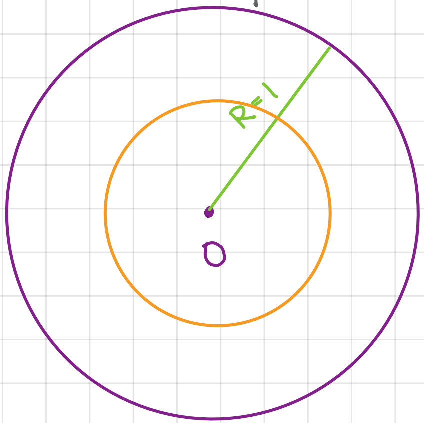
More generally, if $X =$ then

$p_X(k) = P(X=k) =$

We call this distribution

Probability distribution

4) Choose point ω from unit disk uniformly at random

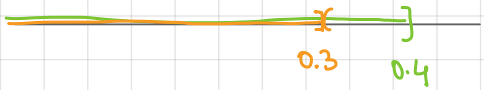


$$X(\omega) = \text{dist}(0, \omega)$$

$$P(X \le r) = \begin{cases} 0, & r < 0 \\ r^2, & 0 \le r \le 1 \\ 1, & r > 1 \end{cases}$$

What can we say about $P(X \in A)$ for other sets $A \subset \mathbb{R}$?

Take $A = (0.3, 0.4]$.



Discrete and continuous random variables

<u>Discrete:</u> There are finitely (or countably) many possible outcomes $\{k_1, k_2, k_3, \dots\}$ for X

μ_X is described by the probability mass function

In this case, by the laws of probability

<u>Continuous:</u> For any real number $t \in \mathbb{R}$,

μ_X is captured by understanding $P(X \leq r)$ as a function of r

For example,

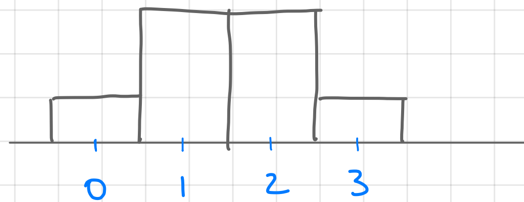
Cumulative Distribution Function (CDF)

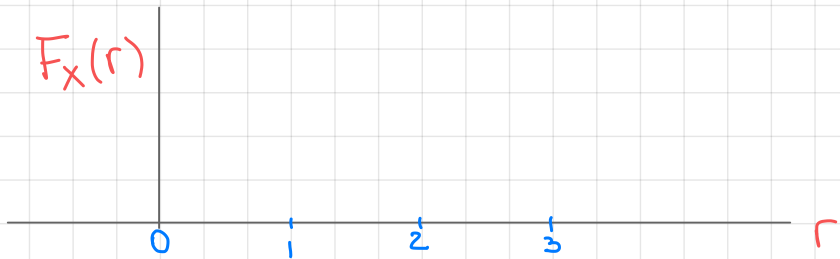
For any random variable X, define

$$F_X(r) = P(X \leq r)$$

K	0	1	2	3
$P_X(K)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Example: $X \sim \text{Bin}\left(3, \frac{1}{2}\right)$





Properties of the CDF

$F_X(r) = P(X \leq r)$

(1) Monotone increasing:

(2) $\lim_{r \to -\infty} F_X(r) = 0$, $\lim_{r \to +\infty} F_X(r) = 1$

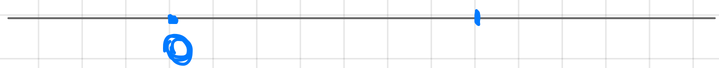
(3) The function F_X is right-continuous:

$$\lim_{t \to r_+} F_X(t) = F_X(r)$$

Corollary: If X is a continuous random variable, F_X is a

Example Shoot an arrow at a circular target of radius 1 (choose point from unit disk uniformly at random)

$$F_X(r) = \begin{cases} 0, & r \leq 0 \\ r^2, & 0 \leq r \leq 1 \\ 1, & r \geq 1 \end{cases}$$



Cumulative distribution function (CDF)

Summary: For any random variable X, $F_X(r) = P(X \leq r)$

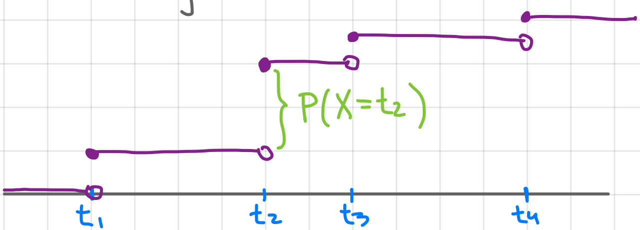
(1) Monotone increasing: $s \leq t \Rightarrow F_X(s) \leq F_X(t)$

(2) $\lim\limits_{r \to -\infty} F_X(r) = 0$, $\lim\limits_{r \to +\infty} F_X(r) = 1$

(3) Right-continuous: $\lim\limits_{t \to r_+} F_X(t) = F_X(r)$

Discrete random variable

Finite or countable set of values with $t_1, t_2, \ldots$, $P(X = t_j) > 0$ and $\sum\limits_j P(X = t_j) = 1$



Continuous random variable

For each real number t, $P(X=t)=0$

Because (1) and (3) this implies that F_X is continuous