MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Cumulative distribution function

Next: ASV 3.3

Week 3:

- no homework this week
- Midterm 1 (Wednesday, February 1, lectures 1-8)

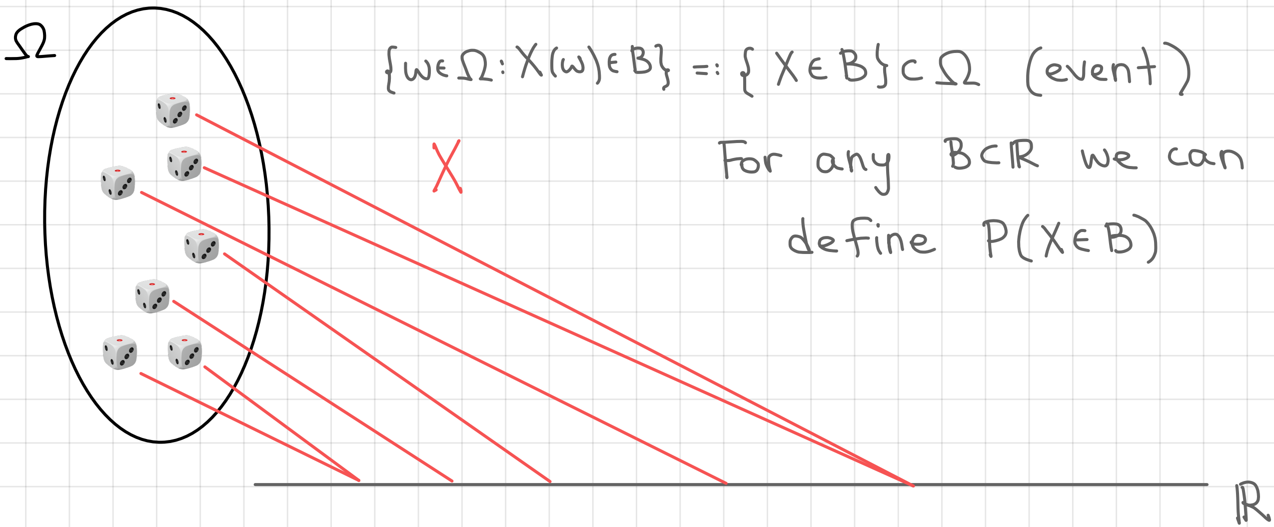
Random variables

$(\Omega, \mathcal{F}, P)$ - probability space

<u>Def</u> A (measurable$^+$) function $X : \Omega \to \mathbb{R}$ is called a **random variable**.

$\{\omega \in \Omega : X(\omega) \in B\} =: \{X \in B\} \subset \Omega$ (event)

For any $B \subset \mathbb{R}$ we can define $P(X \in B)$



Probability distribution

Def Let X be a random variable. The probability distribution of X is the collection of probabilities

$$P(X \in B) \quad \text{for all}^{*} \quad B \subset \mathbb{R}$$

If $(\Omega, \mathcal{F}, P)$ is a probability space, and $X : \Omega \to \mathbb{R}$ is a random variable, we can define a probability measure μ_X on $\mathbb{R}$ given, for any $A \subset \mathbb{R}$, by

$$\mu_X(A) = P(X \in A) = P(\{\omega : X(\omega) \in A\})$$

We call μ_X the probability distribution (or law) of X.

Probability distribution

Toss a fair coin 4 times. Let X = number of tails.

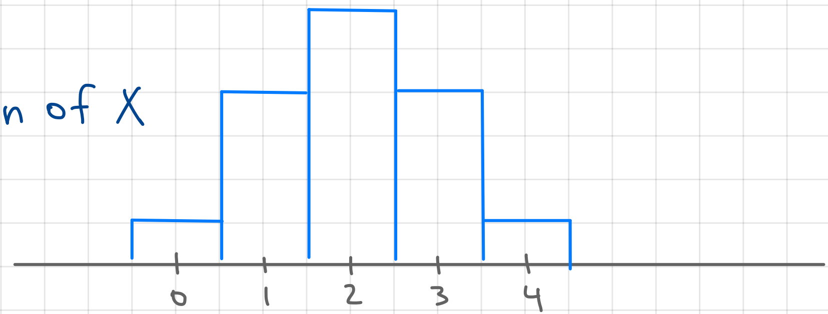
$X \in \{0, 1, 2, 3, 4\}$

$p_X(k) = P(X = k)$

k	0	1	2	3	4
$p_X(k)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$$p_X(k) = \frac{\binom{4}{k}}{2^4}, \quad 0 \le k \le 4$$

probability mass function of X



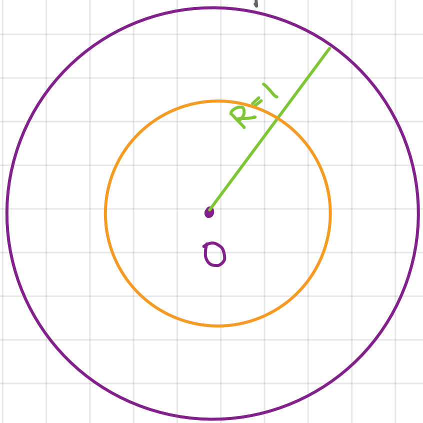
More generally, if X = # tails in n tosses, then

$$p_X(k) = P(X = k) = \frac{1}{2^n}\binom{n}{k}, \quad 0 \le k \le n,$$

We call this distribution Binomial with parameters $n, \frac{1}{2}$

Probability distribution

4) Choose point ω from unit disk uniformly at random



$$X(\omega) = \text{dist}(0, \omega)$$

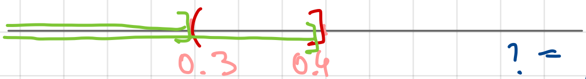
$$P(X \le r) = \begin{cases} 0, & r < 0 \\ r^2, & 0 \le r \le 1 \\ 1, & r > 1 \end{cases}$$

$$\| $$

$$P(X \in (-\infty, r]) = \mu_X((-\infty, r])$$

What can we say about $P(X \in A)$ for other sets $A \subset \mathbb{R}$?

Take $A = (0.3, 0.4]$, $\quad \underbrace{\mu_X((-\infty, 0.4])}_{(0.4)^2} = \underbrace{\mu_X((-\infty, 0.3])}_{(0.3)^2} + \underbrace{\mu_X((0.3, 0.4])}_{(0.4)^2-(0.3)^2 = 0.07}$



$? = \mu_X(\{0.4\})$, $\quad \mu_X((0.4-\varepsilon, 0.4]) = (0.4)^2 - (0.4-\varepsilon)^2$

$= 2 \cdot 0.4\varepsilon - \varepsilon^2 \to 0$ as $\varepsilon \to 0$

For any $\varepsilon > 0$, ok $P(X = 0.4) \le P(X \in (0.4-\varepsilon, 0.4]) \to 0$, $\varepsilon \to 0 \Rightarrow P(X = 0.4) = 0$

Discrete and continuous random variables

<u>Discrete</u>: There are finitely (or countably) many possible outcomes $\{k_1, k_2, k_3, \dots\}$ for X

μ_X is described by the probability mass function

$$p_X(k_i) = P(X = k_i), \quad k \in \{k_1, k_2, \dots\}$$

In this case, by the laws of probability

$$p_X(k) \geq 0 \text{ for each } k, \text{ and } \sum_i p_X(k_i) = 1$$

<u>Continuous</u>: For any real number $t \in \mathbb{R}$, $P(X = t) = 0$

μ_X is captured by understanding $P(X \leq r)$ as a function of r

For example,

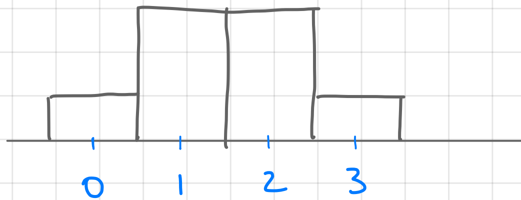
$$P\big(X \in [a,b]\big) = P(\{X = a\} \cup \{X \in (a,b]\})$$

$$= \underbrace{P(X = a)}_{0} + P(X \in (a,b]) = P(X \leq b) - P(X \leq a)$$

Cumulative Distribution Function (CDF)

For any random variable X, define the cumulative distribution function of X $F_X(r) = P(X \le r)$

K	0	1	2	3
$P_X(K)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$



Example: $X \sim \mathrm{Bin}(3, \frac{1}{2})$

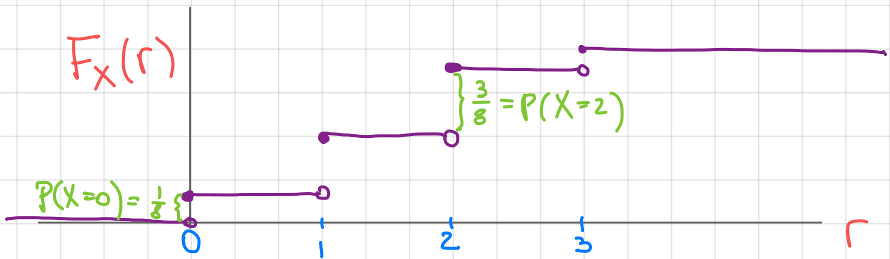
$r<0$, $\{X \le r\} = \emptyset$, $P(X \le r) = 0$

$0 \le r < 1$, $\{X \le r\} = \{X = 0\}$, $P(X \le r) = \frac{1}{8}$

$1 \le r < 2$, $P(X \le r) = P(X=0) + P(X=1) = \frac{1}{2}$

$2 \le r < 3$, $P(X \le r) = P(X=0) + P(X=1) + P(X=2) = \frac{7}{8}$

$r \ge 3$ $P(X \le r) = 1$



Properties of the CDF

$F_X(r) = P(X \le r)$

(1) Monotone increasing: $s < t$, then $F_X(s) \le F_X(t)$

(2) $\lim_{r \to -\infty} F_X(r) = 0$, $\lim_{r \to +\infty} F_X(r) = 1$

(3) The function F_X is right-continuous:

$$\lim_{t \to r_+} F_X(t) = F_X(r)$$

Corollary: If X is a continuous random variable, F_X is a continuous function

Example Shoot an arrow at a circular target of radius 1 (choose point from unit disk uniformly at random)

$$F_X(r) = \begin{cases} 0, & r \le 0 \\ r^2, & 0 \le r \le 1 \\ 1, & r \ge 1 \end{cases}$$