MATH 180A (Lecture A00)

mathweb.ucsd.edu/~ynemish/teaching/180a

Today: Cumulative distribution function

Next: ASV 3.3

Week 3:

- no homework this week
- Midterm 1 (Wednesday, February 1, lectures 1-8)

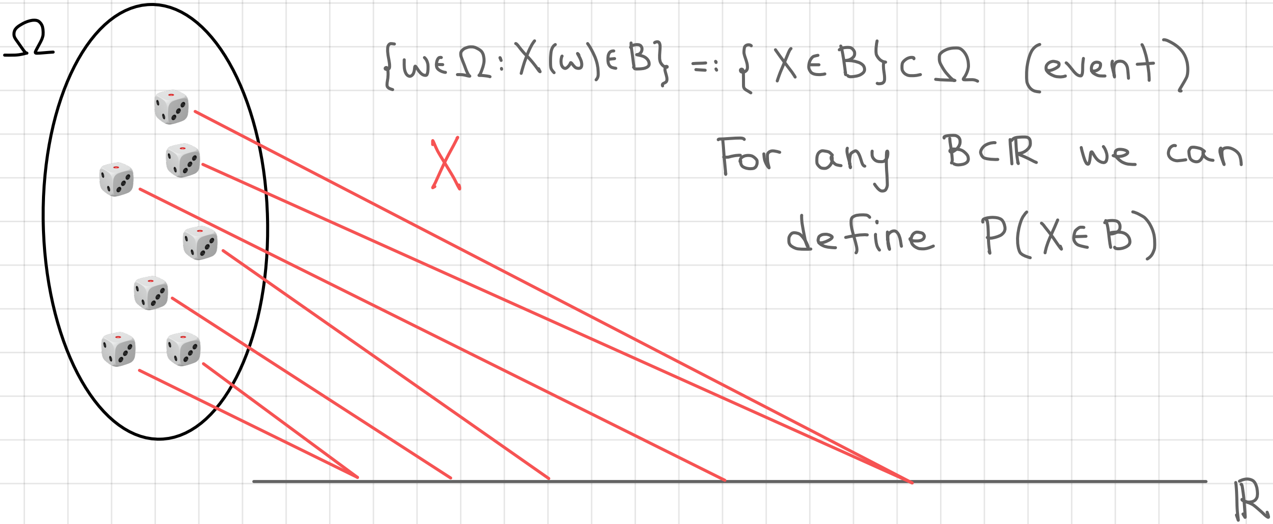
Random variables

$(\Omega, \mathcal{F}, P)$ - probability space

<u>Def</u> A (measurable^{+}) function $X: \Omega \to \mathbb{R}$ is called a random variable.

$\{\omega \in \Omega : X(\omega) \in B\} =: \{X \in B\} \subset \Omega$ (event)

For any $B \subset \mathbb{R}$ we can define $P(X \in B)$



Probability distribution

Def Let X be a random variable. The probability distribution of X is the collection of probabilities

$$P(X \in B) \quad \text{for all}^{*} \quad B \subset \mathbb{R}$$

If $(\Omega, \mathcal{F}, P)$ is a probability space, and $X : \Omega \to \mathbb{R}$ is a random variable, we can define a probability measure μ_X on $\mathbb{R}$ given, for any $A \subset \mathbb{R}$, by

$$\mu_X(A) = P(X \in A) = P(\{\omega : X(\omega) \in A\})$$

We call μ_X the probability distribution (or law) of X.

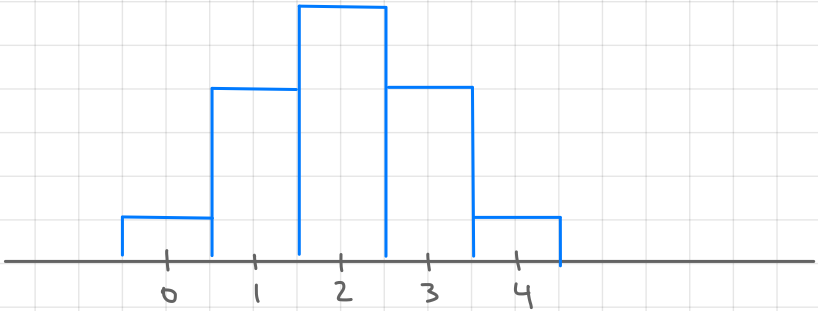
Probability distribution

Toss a fair coin 4 times. Let X = number of tails.

$X \in \{0,1,2,3,4\}$

$p_X(k) = P(X=k)$

k	0	1	2	3	4
$p_X(k)$					



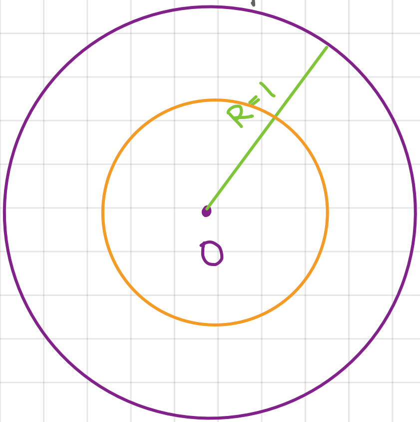
More generally, if $X =$ then

$p_X(k) = P(X=k) =$

We call this distribution

Probability distribution

4) Choose point ω from unit disk uniformly at random



$X(\omega) = \text{dist}(O, \omega)$

$$P(X \le r) = \begin{cases} 0, & r < 0 \\ r^2, & 0 \le r \le 1 \\ 1, & r > 1 \end{cases}$$

What can we say about $P(X \in A)$ for other sets $A \subset \mathbb{R}$?

Take $A = (0.3, 0.4]$,

Discrete and continuous random variables

<u>Discrete:</u> There are finitely (or countably) many possible outcomes $\{k_1, k_2, k_3, \dots\}$ for X

μ_X is described by the probability mass function

$$p_X(k) = P(X = k), \quad k \in \{k_1, k_2, \dots\}$$

In this case, by the laws of probability

<u>Continuous:</u> For any real number $t \in \mathbb{R}$,

μ_X is captured by understanding $P(X \leq r)$ as a function of r

For example,

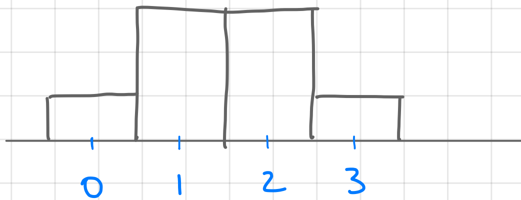
Cumulative Distribution Function (CDF)

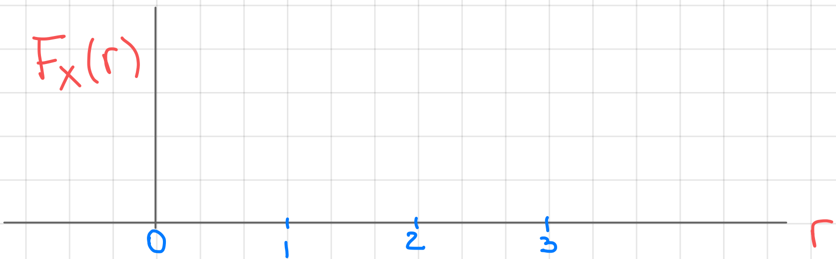
For any random variable X, define

$$F_X(r) = P(X \le r)$$

K	0	1	2	3
$P_X(k)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Example: $X \sim \text{Bin}\left(3, \frac{1}{2}\right)$





Properties of the CDF

$F_X(r) = P(X \le r)$

(1) Monotone increasing :

(2) $\lim_{r \to -\infty} F_X(r) = 0$, $\lim_{r \to +\infty} F_X(r) = 1$

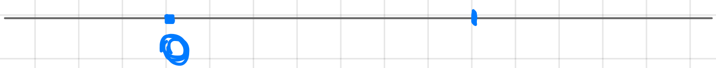
(3) The function F_X is right-continuous :

$$\lim_{t \to r_+} F_X(t) = F_X(r)$$

Corollary: If X is a continuous random variable, F_X is a

Example Shoot an arrow at a circular target of radius 1 (choose point from unit disk uniformly at random)

$$F_X(r) = \begin{cases} 0, & r \le 0 \\ r^2, & 0 \le r \le 1 \\ 1, & r \ge 1 \end{cases}$$



Cumulative distribution function (CDF)

Summary: For any random variable X, $F_X(r) = P(X \leq r)$

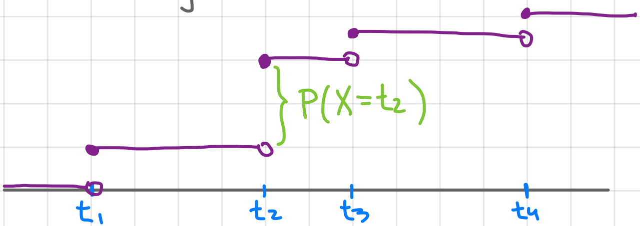
(1) Monotone increasing: $s \leq t \Rightarrow F_X(s) \leq F_X(t)$

(2) $\lim_{r \to -\infty} F_X(r) = 0$, $\lim_{r \to +\infty} F_X(r) = 1$

(3) Right-continuous: $\lim_{t \to r_+} F_X(t) = F_X(r)$

Discrete random variable

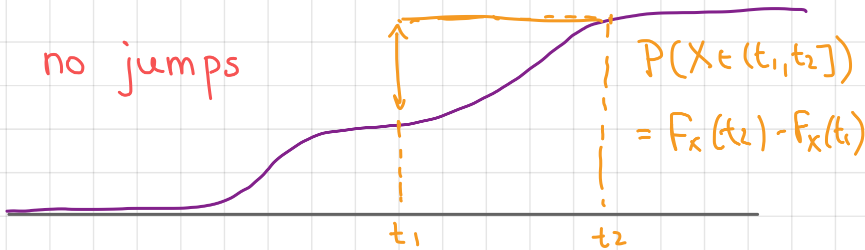
Finite or countable set of values with $t_1, t_2, \ldots$, $P(X=t_j) > 0$ and $\sum_j P(X=t_j) = 1$



Continuous random variable

For each real number t, $P(X=t) = 0$

Because (1) and (3) this implies that F_X is continuous



Densities (PDF)

Some continuous random variables have probability densities. This is the infinitesimal version of the probability mass function.

X discrete, $X \in \{t_1, t_2, \dots\}$

$p_X(t) = P(X=t)$

probability mass function

X continuous

$P(X=t) = 0$ for all $t \in \mathbb{R}$

Densities (PDF)

Example Shoot an arrow at a circular target of radius 1.

X = distance from center

$$F_X(r) = \begin{cases} 0, & r \leq 0 \\ r^2, & 0 \leq r \leq 1 \\ 1, & r \geq 1 \end{cases}$$

PDF: existence

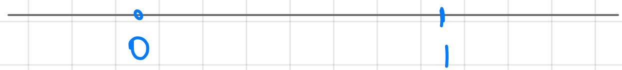
Thm: If F_X is continuous and (piecewise) differentiable, then X has density

Proof: Follows from FTC ∎

Example Let X = random number chosen uniformly on $[0,1]$

We have seen that in this case $P(X \in [s,t]) = t-s$, $0 \le s < t \le 1$

$$F_X(r) = P(X \le r) = \begin{cases} \end{cases}$$



$$f_X(r) =$$

PDF

Example Let $f(t) = \begin{cases} c\sqrt{1-t^2}, & |t| \le 1 \\ 0, & \text{otherwise} \end{cases}$, $c > 0$

Q: (When) Is $f(t)$ a PDF of some random variable?

- $f \ge 0$
- $1 = \int_{-\infty}^{+\infty} f(t)\,dt =$

f is a PDF

Question

Your car is in a minor accident. The damage repair cost is a random number between 100 and 1500 dollars. Your insurance deductible is 500 dollars.

Z = your out of pocket expenses

Question: The random variable Z is

(a) continuous

(b) discrete

(c) neither

(d) both