Name (last, first): ______________________________

Student ID: ______________________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **The exam consists of 8 questions. Your answers must be carefully justified to receive credit.**

☐ **This exam will be scanned. Make sure you write ALL SOLUTIONS on the paper provided. DO NOT REMOVE ANY OF THE PAGES.**

☐ **No calculators, phones, or other electronic devices are allowed.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You are allowed to use two 8.5 by 11 inch sheets of paper with hand-written notes (on both sides); no other notes (or books) are allowed.**

1. (10 points) You have an urn that initially contains 6 red balls, 2 black balls and 1 green ball. On the first step, you choose one ball uniformly at random from the urn, look at its color, and then return it back to the urn together with one more ball of the same color (e.g., if you pick a red ball, then you put it back to the urn together with another red ball). Then on the second step you choose a ball uniformly at random from the urn (note that on the second step the urn contains the additional ball).

   What is the probability that on the second step you choose a red ball?

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

2. (10 points) Every morning Frank chooses how to commute to work: by car or by bicycle. He chooses bicycle with probability 0.7. The probability that Frank is late to work if he rides a bicycle is 0.1, and the probability that he is late if he drives a car is 0.2. Frank is late today. What is the probability, that he came to work by bicycle?

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

3. (10 points) Consider a point $P = (X, Y)$ chosen uniformly at random inside the rectangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 1)$, $(2, 0)$ and $(2, 1)$. Let $Z = \max(X, Y)$ be the random variable defined as the maximum of the two coordinates of the point. [**Hint**. Draw a picture.]

   (a) Compute and plot the cumulative distribution function of Z.

   (b) Determine if Z is continuous, discrete or neither. If continuous, determine the probability density function of Z. If discrete, determine the probability mass function of Z. If neither, explain why.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

4. (10 points) A study showed that 2% of San Diego residents own a boat.

   (a) Estimate the probability that among 100 randomly interviewed San Diego residents there are at least 3 boat owners.

   (b) Explain why the approximation that you used in part (a) is better compared to other approximations that you know. [For full credit, present your answer in the closed form (not as an infinite series); you may leave your answer in terms of e^x or $\Phi(x)$]

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

5. (10 points) Let X and Y be independent random variables uniformly distributed on the interval $[0, 1]$, i.e., $X \sim \mathcal{U}[0, 1]$, $Y \sim \mathcal{U}[0, 1]$.

   (a) Compute the moment generating function of the sum $X + Y$.

   (b) Show that for any $t \in \mathbb{R}$

$$(e^t - 1)^2 = e^{2t} - 2e^t + 1 = \sum_{k=2}^{\infty} \frac{2^k}{k!} t^k - \sum_{k=2}^{\infty} \frac{2}{k!} t^k. \tag{1}$$

   (c) Use the results of (a) and (b) to compute $E((X + Y)^n)$, moments of the sum, for any $n \in \mathbb{N}$.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

6. (10 points) Suppose that $X \sim \text{Geom}(p)$ and $Y \sim \text{Geom}(q)$ are independent random variables. Find the probability $P(X < Y)$.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

7. (10 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables with $X_1 \sim \text{Unif}[0,1]$. Let $Y = \min(X_1, \ldots, X_n)$. Find the CDF F_Y and density f_Y.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

8. (10 points) Let T be the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$ (including the interior). Suppose that $P = (X, Y)$ is a point chosen uniformly at random inside of T.

   (a) What is the joint density function of (X, Y)? Use this to compute $\mathrm{Cov}(X, Y)$.

   (b) Determine if X and Y are independent.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)