Name (last, first): ______________________________

Student ID: ______________________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **The exam consists of 4 questions. Your answers must be carefully justified to receive credit.**

☐ **This exam will be scanned. Make sure you write ALL SOLUTIONS on the paper provided. DO NOT REMOVE ANY OF THE PAGES.**

☐ **No calculators, phones, or other electronic devices are allowed.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

   (a) Suppose that $A, B \in \mathcal{F}$ satisfy
   $$P(A) + P(B) > 1.$$
   Making no further assumptions on A and B, prove that $A \cap B \neq \emptyset$.

   Solution. First method (proof by contradiction). Assume that $A \cap B = \emptyset$. Then $P(A \cup B) = P(A) + P(B) > 1$, contradiction. Therefore, $A \cap B \neq 0$.
   Second method.
   $$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1,$$
   therefore $P(A \cap B) \geq P(A) + P(B) - 1 > 0$, and we conclude that $A \cap B \neq \emptyset$.

   (b) Prove that A is independent from itself if and only if $P(A) \in \{0, 1\}$.

   Solution. By definition, events A and B are independent if $P(A \cap B) = P(A)P(B)$. Take $A = B$, so that A being independent of A is equivalent to $P(A \cap A) = P(A)P(A)$. Since $A \cap A = A$, we have that $P(A)$ satisfies
   $$(P(A))^2 = P(A).$$
   The only two numbers that satisfy the above equation are numbers 0 and 1.

2. An urn contains 2 white balls and 4 black balls. You remove the balls one by one from the urn (without replacement).

   (a) What is the probability that the first two balls removed from the urn are black?

   Solution. Suppose that we choose two balls from the urn, and let A be the event that two balls are black. Since the balls are chosen without replacement, order does not matter, we have

$$P(A) = \frac{\binom{4}{2}}{\binom{6}{2}} = \frac{2}{5}. \tag{1}$$

   (b) What is the probability that the last removed ball is white?

   Solution. Suppose that we remove all balls from the urn, and let B be the event that the last ball is white. Then

$$\#\Omega = 6!, \quad \#B = 5! \cdot 2, \quad P(B) = \frac{5! \cdot 2}{6!} = \frac{1}{3}. \tag{2}$$

3. A box contains 3 coins, two of which are fair and the third has probability 3/4 of coming up heads. A coin is chosen randomly from the box and tossed 3 times.

   (a) What is the probability that all 3 tosses are heads?

   Solution. Define the following events:

$$\begin{aligned} A &= \{\text{all 3 tosses are heads}\} \\ B_1 &= \{\text{chosen coin is fair}\} \\ B_2 &= \{\text{chosen coin is biased}\}. \end{aligned}$$

   Then B_1 and B_2 form a partition of the sample space with

$$P(B_1) = \frac{2}{3}, \qquad P(B_2) = \frac{1}{3},$$

   and depending on which coin was chosen from the box, we have the conditional probabilities of observing heads on all three tosses

$$P(A|B_1) = \left(\frac{1}{2}\right)^3, \qquad P(A|B_2) = \left(\frac{3}{4}\right)^3.$$

   Then we can compute $P(A)$ using the law of total probability

$$\begin{aligned} P(A) &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) \\ &= \frac{1}{8} \cdot \frac{2}{3} + \left(\frac{3}{4}\right)^3 \frac{1}{3} \\ &= \frac{43}{192}. \end{aligned}$$

   (b) Given that all three tosses are heads, what is the probability that the biased coin was chosen?

   Solution. Using the Bayes' rule, we have

$$P(B_2|A) = \frac{P(A|B_2)P(B_2)}{P(A)} = \frac{27}{43}.$$

4. You roll three fair *four-sided* dice.

   (a) Compute the probability that there will be at least one four, *given* that all three dice give different numbers.

   Solution. Denote by A the event that there will be at least one four and by B the event that all three dice give different numbers. If Ω is the sample space, then $\#\Omega = 4^3$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \tag{3}$$

   $A \cap B$ is the event that all three numbers are different, and one of the numbers is four. Then $\#(A \cap B) = 3 \cdot 3 \cdot 2$ and $\#B = 4 \cdot 3 \cdot 2$. Therefore

$$P(A|B) = \frac{3 \cdot 3 \cdot 2}{4 \cdot 3 \cdot 2} = \frac{3}{4}. \tag{4}$$

   (b) Compute the (unconditional) probability that there will be at least one four. [**Hint.** Use complement]

   Solution.
   Without conditioning on B, $P(A) = 1 - P(A^C)$, where A^C is the event that none of the dice gives four. Therefore, $\#A^C = 3^3$, and thus

$$P(A) = 1 - P(A^C) = 1 - \frac{3^3}{4^3} = \frac{64 - 27}{64} = \frac{37}{64}. \tag{5}$$

   Note that $P(A|B) > P(A)$.