Name (last, first): ____________________

Student ID: ____________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **The exam consists of 4 questions. Your answers must be carefully justified to receive credit.**

☐ **This exam will be scanned. Make sure you write ALL SOLUTIONS on the paper provided. DO NOT REMOVE ANY OF THE PAGES.**

☐ **No calculators, phones, or other electronic devices are allowed.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with hand-written notes (on both sides); no other notes (or books) are allowed.**

1. (20 points) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose that A and B are events in $\mathcal{F}$ such that $P(A) = 0.5$ and $P(B) = 0.8$. What possible range of values can $P(A \cap B)$ have?

Solution. From the monotonicity of the probability measure we have

$$P(A \cap B) \leq \min\{P(A), P(B)\} = 0.5.$$

From the inclusion-exclusion formula for two events we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 = 0.3,$$

where we used that

$$P(A \cup B) \leq 1.$$

We conclude that

$$0.3 \leq P(A \cap B) \leq 0.5,$$

i.e., the possible range of values of $P(A \cap B)$ is the interval $[0.3, 0.5]$.

2. (20 points) Suppose that we have an urn with 5 red balls, 3 blue balls and 2 green balls. We draw a ball uniformly at random, record its color, and **do not** put the ball back into the urn. We repeat this 3 times.

   (a) What is the probability that the first ball is red and the second ball is blue?

   Solution. We can view this experiment as sampling 3 balls out of 10 without replacement (order matters), therefore, the sample space Ω has the size

$$\#\Omega = 10 \cdot 9 \cdot 8.$$

   Let A be the event that the first ball is red and the second ball is blue. Then

$$\#A = 5 \cdot 3 \cdot 8$$

   (5 ways to choose the first red ball, 3 ways to choose the second blue ball, and once the first two balls are chosen there are 8 ways to choose the last ball). Thus

$$P(A) = \frac{5 \cdot 3 \cdot 8}{10 \cdot 9 \cdot 8} = \frac{1}{6}.$$

   (b) What is the probability that all three sampled balls **do not** have the same color?

   Solution. Let D be the event that all three do not have the same color. We can compute the probability of D by computing the probability of the complement of D

$$P(D) = 1 - P(D^C),$$

   where D^C is the event that all three balls have the same color. The even D^C can be rewritten as

$$D^C = R \cup B \cup G,$$

   where R is the event that all three balls are red, B is the event that all three balls are blue, and G is the event that all three balls are green. Then R, B and G are disjoint, and

$$\#R = 5 \cdot 4 \cdot 3, \quad \#B = 3 \cdot 2 \cdot 1, \quad \#G = 0.$$

   Therefore,

$$P(D^C) = \frac{5 \cdot 4 \cdot 3 + 3 \cdot 2 \cdot 1}{10 \cdot 9 \cdot 8} = \frac{1}{12} + \frac{1}{120} = \frac{11}{120},$$

   and

$$P(D) = 1 - \frac{11}{120} = \frac{109}{120}.$$

3. (20 points) An insurance company has two types of customers, careful and reckless. We know that with probability 0.99 a careful customer does not have an accident during a year. On the other hand, a reckless customer has an accident during a year with probability 0.81.

   90% of the customers are careful and 10% of the customers are reckless.

   (a) What is the probability that a randomly chosen customer has an accident this year?

   Solution. Denote the events $A = \{\text{the randomly chosen customer has an accident}\}$ and $C = \{\text{the randomly chosen person is careful}\}$. Then the events C and C^C form a partition of the sample space, and thus we can compute $P(A)$ using the law of total probability

$$P(A) = P(A|C)P(C) + P(A|C^C)P(C^C). \tag{1}$$

   It is given that

$$P(C) = 0.9, \quad P(C^C) = 0.1.$$

   It is also given that

$$P(A^C|C) = 0.99, \quad P(A|C^C) = 0.81.$$

   Since the conditional probability $P(\,\cdot\,|C)$ is a probability measure, we have

$$P(A|C) = 1 - P(A^C|C) = 0.01.$$

   After plugging all the information into formula (1) we find

$$P(A) = 0.01 \cdot 0.9 + 0.81 \cdot 0.1 = 0.009 + 0.081 = 0.09 = 9\%.$$

   (b) Suppose a randomly chosen customer has an accident this year. What is the probability that this customer is one of the careful customers?

   Solution. We have to compute the conditional probability $P(C|A)$. From the Bayes' rule we find

$$P(C|A) = \frac{P(A|C)P(C)}{P(A)} = \frac{0.01 \cdot 0.9}{0.09} = 0.1 = 10\%.$$

4. (20 points) You perform the following two-step experiment. On the first step, choose a number R from the set $\{1, 3, 5\}$ uniformly at random. On the second step, choose a point uniformly at random from the disk of radius R, and denote by X the distance from the point to the center of the disk.

   (a) Compute $P(X \leq 2)$.

   Solution. The collection of events $\{R = 1\}$, $\{R = 3\}$ and $\{R = 5\}$ form the partition of the sample space. Since the number R is chosen uniformly at random, we have

$$P(R = i) = \frac{1}{3}$$

   for $i \in \{1, 3, 5\}$. Now we can use the law of total probability to compute $P(X \leq 2)$

$$\begin{aligned} &P(X \leq 2) \\ &\quad = P(X \leq 2|R = 1)P(R = 1) + P(X \leq 2|R = 3)P(R = 3) + P(X \leq 2|R = 5)P(R = 5). \end{aligned}$$

   Notice that

$$P(X \leq 2|R = 1) = 1, \quad P(X \leq 2|R = 3) = \frac{2^2}{3^2}, \quad P(X \leq 2|R = 5) = \frac{2^2}{5^2}.$$

   Therefore,

$$P(X \leq 2) = 1 \cdot \frac{1}{3} + \frac{2^2}{3^2} \cdot \frac{1}{3} + \frac{2^2}{5^2} \cdot \frac{1}{3} = \frac{1}{3}\Big(1 + \frac{2^2}{3^2} + \frac{2^2}{5^2}\Big).$$

   (b) Compute $P(X \leq t)$ for all real numbers $t \in \mathbb{R}$. Express your answer as a function of t

$$F(t) = P(X \leq t).$$

   Solution. Repeating the above computations for general $t \in \mathbb{R}$ we obtain

$$P(X \leq t) = \begin{cases} 0, & t < 0, \\ \frac{1}{3}\Big(\frac{t^2}{1^2} + \frac{t^2}{3^2} + \frac{t^2}{5^2}\Big), & 0 \leq t < 1, \\ \frac{1}{3}\Big(1 + \frac{t^2}{3^2} + \frac{t^2}{5^2}\Big), & 1 \leq t < 3, \\ \frac{1}{3}\Big(1 + 1 + \frac{t^2}{5^2}\Big), & 3 \leq t < 5, \\ 1, & t \geq 5. \end{cases}$$