MATH 180C HOMEWORK 8

SPRING 2023

Due date: **Friday 6/9/2023 11:59 PM** (via Gradescope)
Note that there are *Exercises* and *Problems* in the textbook. Make sure you read the homework carefully to find the assigned question.

1. *Pinsky and Karlin, Exercise 8.1.2.* Let $\{B(t); t \geq 0\}$ be a standard Brownian motion and $c > 0$ a constant. Show that the process defined by $W(t) = cB(t/c^2)$ is a standard Brownian motion.

   Solution. Denote $X(t) := cB(t/c^2)$. Consider the function $g : [0, +\infty) \to [0, +\infty)$ given by $g(x) = x/c^2$. Function g is a continuous bijection from $[0, +\infty)$ to $[0, +\infty)$, therefore, if B is continuous, then $B(g(t)) = B(t/c^2)$ is continuous on $[0, +\infty)$, and thus $cB(t/c^2)$ is also continuous on $[0, +\infty)$. Since $P(B(t)$ is continuous $) = 1$, we conclude that $P(X(t)$ is continuous $) = 1$, with probability 1 the process $X(t)$ has continuous trajectories.

   Next, fix any $0 \leq t_1 < t_2 < \cdots < t_n$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Then
$$\sum_{i=1}^{n} \alpha_i X(t_i) = \sum_{i=1}^{n} \alpha_i c B(t_i/c^2)$$
   has normal distribution (since $B(t)$ is Gaussian). We conclude that $X(t)$ is a Gaussian process with continuous sample paths.

   Finally, compute the mean and the covariance function of $X(t)$. It is easy to see that for any $t \geq 0$
$$E(X(t)) = E(cB(t/c^2)) = 0.$$
   For the covariance function we have
$$\begin{aligned}\operatorname{Cov}(X(s), X(t)) &= \operatorname{Cov}(cB(s/c^2), cB(t/c^2)) \\ &= c^2 \operatorname{Cov}(B(s/c^2), B(t/c^2)) \\ &= c^2 \min\{s/c^2, t/c^2\} \\ &= \min\{s, t\}.\end{aligned}$$
   We see that $(X(t))_{t\geq 0}$ is a Gaussian stochastic process with continuous sample paths, mean 0 and covariance function $\Gamma(s, t) = \min\{s, t\}$. Thus, $(X(t))_{t\geq 0}$ is a Brownian motion.

2. *Pinsky and Karlin, Exercise 8.1.3.* (a) Show that
$$\frac{d\phi(x)}{dx} = \phi'(x) = -x\phi(x),$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

(b) Use the result in (a) together with the chain rule of differentiation to show that

$$p(y,t|x) = \phi_t(y-x) = \frac{1}{\sqrt{t}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big)$$

satisfies the diffusion equation

(1) $$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2}.$$

Solution. (a) We use direct computation

$$\phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}(-x) = -x\phi(x).$$

(b) We have to verify that $p(y,t|x) = \frac{1}{\sqrt{t}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big)$ satisfies (1) with $\sigma^2 = 1$. First we compute the partial derivative with respect to t

$$\frac{\partial}{\partial t}\bigg(\frac{1}{\sqrt{t}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big)\bigg) = -\frac{1}{2t^{3/2}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big) + \frac{1}{\sqrt{t}}\Big(-\frac{y-x}{\sqrt{t}}\Big)\phi\Big(\frac{y-x}{\sqrt{t}}\Big)\Big(-\frac{y-x}{2t^{3/2}}\Big)$$
$$= -\frac{1}{2t^{3/2}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big)\bigg(1 - \frac{(y-x)^2}{t}\bigg).$$

Now compute the second derivative with respect to x

$$\frac{\partial}{\partial x}\bigg(\frac{1}{\sqrt{t}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big)\bigg) = -\frac{1}{\sqrt{t}}\Big(\frac{y-x}{\sqrt{t}}\Big)\phi\Big(\frac{y-x}{\sqrt{t}}\Big)\Big(-\frac{1}{\sqrt{t}}\Big)$$
$$= \frac{y-x}{t^{3/2}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big),$$

and

$$\frac{\partial^2}{\partial x^2}\bigg(\frac{1}{\sqrt{t}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big)\bigg) = -\frac{1}{t^{3/2}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big) + \frac{y-x}{t^{3/2}}\frac{\partial}{\partial x}\bigg(\phi\Big(\frac{y-x}{\sqrt{t}}\Big)\bigg)$$
$$= -\frac{1}{t^{3/2}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big) + \frac{y-x}{t^{3/2}}\frac{y-x}{t^{3/2}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big)$$
$$= -\frac{1}{t^{3/2}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big)\bigg(1 - \frac{(y-x)^2}{t^{3/2}}\bigg).$$

We see that

$$\frac{\partial}{\partial t}\bigg(\frac{1}{\sqrt{t}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big)\bigg) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\bigg(\frac{1}{\sqrt{t}}\phi\Big(\frac{y-x}{\sqrt{t}}\Big)\bigg).$$

3. *Pinsky and Karlin, Exercise 8.1.4.* Consider a standard Brownian motion $\{B(t); t \geq 0\}$ at times $0 < u < u+v < u+v+w$, where $u, v, w > 0$.
   (a) Evaluate the product moment $E(B(u)B(u+v)B(u+v+w))$.
   (b) Evaluate the product moment $E(B(u)B(u+v)B(u+v+w)B(u+v+w+x))$, where $x > 0$.

Solution. (a) We represent $B(u+v+w)$ as a sum of independent Gaussian random variables

$$\begin{aligned}
&E(B(u)B(u+v)B(u+v+\\
&\quad = E\Big(B(u)B(u+v)\big(B(u+v) + (B(u+v+w) - B(u+v))\big)\Big)\\
&\quad = E\Big(B(u)B(u+v)B(u+v)\Big) + E\Big(B(u)B(u+v)\big(B(u+v+w) - B(u+v)\big)\Big)\\
&\quad = E\Big(B(u)(B(u+v))^2\Big),
\end{aligned}$$

where we used that $B(u)B(u+v)$ and $B(u+v+w) - B(u+v)$ are independent and $E(B(u+v+w) - B(u+v)) = 0$. Next we represent $B(u+v)$ as a sum of independent Gaussian random variables

$$\begin{aligned}
&E\Big(B(u)(B(u+v))^2\Big) = E\Big(B(u)\big(B(u) + B(u+v) - B(u)\big)^2\Big)\\
&\quad = E\Big(B(u)\big((B(u))^2 + 2B(u)(B(u+v) - B(u)) + (B(u+v) - B(u))^2\big)\Big)\\
&\quad = E\Big((B(u))^3\Big) + 2E\Big((B(u))^2(B(u+v) - B(u))\Big) + E\Big(B(u)(B(u+v) - B(u))^2\Big)\\
&\quad = 0 + 0 + 0 = 0.
\end{aligned}$$

(b) Now we do the same with $E(B(u)B(u+v)B(u+v+w)B(u+v+w+x))$

$$\begin{aligned}
&E(B(u)B(u+v)B(u+v+w)B(u+v+w+x))\\
&\quad = E\Big(B(u)B(u+v)(B(u+v+w))^2\Big)\\
&\quad = E\Big(B(u)B(u+v)(B(u+v) + B(u+v+w) - B(u+v))^2\Big)\\
&\quad = E\Big(B(u)\big(B(u+v)\big)^3\Big) + E\Big(B(u)B(u+v)\big(B(u+v+w) - B(u+v)\big)^2\Big)\\
&\quad = E\Big(B(u)\big(B(u) + B(u+v) - B(u)\big)^3\Big) + wE\Big(B(u)\big(B(u) + B(u+v) - B(u)\big)\Big)\\
&\quad = E\Big((B(u))^4\Big) + 3E\Big((B(u))^2(B(u+v) - B(u))^2\Big) + uw\\
&\quad = 3u^2 + 3uv + uw = u(3u + 3v + w).
\end{aligned}$$

4. *Pinsky and Karlin, Exercise 8.1.5.* Determine the covariance functions for the stochastic processes
(a) $U(t) = e^{-t}B(e^{2t})$, for $t \geq 0$
(b) $V(t) = (1-t)B(t/(1-t))$, for $0 \leq t \leq 1$
(c) $W(t) = tB(1/t)$, with $W(0) = 0$.
$B(t)$ is standard Brownian motion.

Solution. (a)

$$\begin{aligned}\mathrm{Cov}(U(s),U(t)) &= \mathrm{Cov}(e^{-s}B(e^{2s}),e^{-t}B(e^{2t}))\\ &= e^{-(s+t)}\mathrm{Cov}(B(e^{2s}),B(e^{2t}))\\ &= e^{-(s+t)}\min\{e^{2s},e^{2t}\}\\ &= e^{-(s+t-2\min\{s,t\})} = e^{-(\max\{s,t\}-\min\{s,t\})} = e^{-|s-t|}.\end{aligned}$$

(b)

$$\begin{aligned}\mathrm{Cov}(V(s),V(t)) &= \mathrm{Cov}\Big((1-s)B(s/(1-s)),(1-t)B(t/(1-t))\Big)\\ &= (1-s)(1-t)\mathrm{Cov}\Big(B(s/(1-s)),B(t/(1-t))\Big)\\ &= (1-s)(1-t)\min\{s/(1-s),t/(1-t)\}\\ &= (1-s)(1-t)\min\{s,t\}/(1-\min\{s,t\})\\ &= (1-\max\{s,t\})\min\{s,t\} = \min\{s,t\}-\max\{s,t\}\min\{s,t\}\\ &= \min\{s,t\}-st.\end{aligned}$$

(c)

$$\begin{aligned}\mathrm{Cov}(W(s),W(t)) &= \mathrm{Cov}(sB(1/s),tB(1/t))\\ &= st\mathrm{Cov}(B(1/s),B(1/t))\\ &= st\min\{1/s,1/t\}\\ &= st/\max\{s,t\} = \min\{s,t\}.\end{aligned}$$

5. *Pinsky and Karlin, Exercise 8.1.6.* Consider a standard Brownian motion $\{B(t);t\geq 0\}$ at times $0<u<u+v<u+v+w$, where $u,v,w>0$.
 (a) What is the probability distribution of $B(u)+B(u+v)$?
 (b) What is the probability distribution of $B(u)+B(u+v)+B(u+v+w)$?

 Solution. Since $B(t)$ is a Gaussian process, $B(u)+B(u+v)$ and $B(u)+B(u+v)+B(u+v+w)$ have Gaussian distribution. It is easy to see that both random variables have zero expectation. It is enough to compute the variance of each of these random variables.

 (a)

$$\begin{aligned}&\mathrm{Var}\Big(B(u)+B(u+v)\Big)\\ &= \mathrm{Cov}\Big(B(u)+B(u+v),B(u)+B(u+v)\Big)\\ &= \mathrm{Cov}\Big(B(u),B(u)\Big)+2\mathrm{Cov}\Big(B(u),B(u+v)\Big)+\mathrm{Cov}\Big(B(u+v),B(u+v)\Big)\\ &= u+2u+(u+v) = 4u+v,\end{aligned}$$

thus $B(u)+B(u+v)\sim N(0,4u+v)$.

(b)

$$\begin{aligned}
&\mathrm{Var}\Big(B(u)+B(u+v)+B(u+v+w)\Big)\\
&\quad=\mathrm{Cov}\Big(B(u)+B(u+v)+B(u+v+w),B(u)+B(u+v)+B(u+v+w)\Big)\\
&\quad=\mathrm{Cov}\Big(B(u),B(u)\Big)+2\mathrm{Cov}\Big(B(u),B(u+v)\Big)+2\mathrm{Cov}\Big(B(u),B(u+v+w)\Big)\\
&\qquad\quad+\mathrm{Cov}\Big(B(u+v),B(u+v)\Big)+2\mathrm{Cov}\Big(B(u+v),B(u+v+w)\Big)\\
&\qquad\quad+\mathrm{Cov}\Big(B(u+v+w),B(u+v+w)\Big)\\
&\quad=u+2u+2u+(u+v)+2(u+v)+(u+v+w)=9u+4v+w,
\end{aligned}$$

thus $B(u)+B(u+v)+B(u+v+w)\sim N(0,9u+4v+w)$.

6. *Pinsky and Karlin, Problem 8.1.4.* Let $\alpha_1,\dots,\alpha_n$ be real constants. Argue that

$$\sum_{i=1}^{n}\alpha_i B(t_i)$$

is normally distributed with mean zero and variance

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\min\{t_i,t_j\}.$$

Solution. Denote $Z:=\sum_{i=1}^{n}\alpha_i B(t_i)$. Since $B(t)$ is a Gaussian process, Z has normal distribution. Now compute the mean and the variance of Z:

$$E(Z)=E\Big(\sum_{i=1}^{n}\alpha_i B(t_i)\Big)=\sum_{i=1}^{n}\alpha_i E\Big(B(t_i)\Big)=0,$$

and

$$\begin{aligned}
\mathrm{Var}(Z)&=\mathrm{Var}\Big(\sum_{i=1}^{n}\alpha_i B(t_i)\Big)\\
&=\mathrm{Cov}\Big(\sum_{i=1}^{n}\alpha_i B(t_i),\sum_{j=1}^{n}\alpha_j B(t_j)\Big)\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\mathrm{Cov}\Big(B(t_i),B(t_j)\Big)\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\min\{t_i,t_j\}.
\end{aligned}$$