MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Stationary distribution and long-run behavior of CTMC

Next: PK 2.4

Week 4:

- HW3 due Saturday, April 29 on Gradescope
- Midterm 1: Friday, April 28

Long run behavior of discrete time MC. Summary

Let $(X_n)_{n\geq 0}$ be a disrete time MC on $\{0,\dots,N\}$ with stationary transition probability matrix $P=(P_{ij})_{i,j=0}^{N}$.

- P is called regular if there exists k such that $[P^k]_{ij}>0$ for all i,j. [P is regular iff (X_n) is irreducible and aperiodic]

Thm. If P is regular, then there exist $\pi_0,\dots,\pi_N\in\mathbb{R}$ s.t.

1) $\pi_i>0 \quad \forall i$
2) $\sum_{i=0}^{N}\pi_i=1$
3) $\forall j \quad \lim_{n\to\infty} P_{ij}^{(n)}=\pi_j$

$(\pi_0,\dots,\pi_n)$ is called limiting (stationary) distribution of (X_n)

$(\pi_0,\dots,\pi_n)$ is uniquely defined by the system of equations

$$\begin{cases}\pi_j=\sum_{i=0}^{N}\pi_i P_{ij}, \\ \sum_{i=0}^{N}\pi_i=1\end{cases}$$

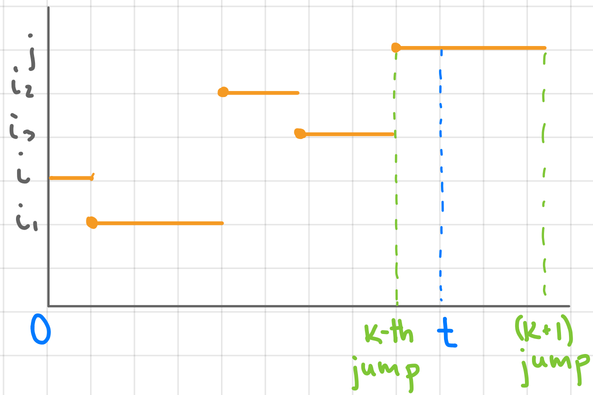
$$(\pi_0,\pi_1,\dots,\pi_N)=(\pi_0,\dots,\pi_N)P$$

Long run behavior of continuous time MC.

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0, \dots, N\}$ and let $(Y_n)_{n\geq 0}$ be the embedded jump chain.

<u>Def</u>. $(X_t)_{t\geq 0}$ is called irreducible if its jump chain $(Y_n)_{n\geq 0}$ is irreducible (consisting of one communicating class)

<u>Thm</u>. If $(X_t)_{t\geq 0}$ is irreducible, then



Idea of the proof:

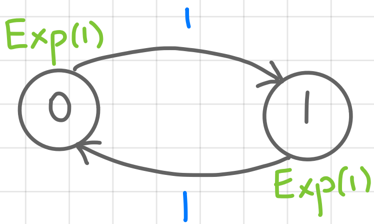
- Y_n is irreducible $\Rightarrow \exists\ i_1, \dots, i_{k-1}$ s.t.
  $P(Y_k = j, Y_{k-1} = i_{k-1}, \dots, Y_1 = i_1 \mid Y_0 = i) > 0$
- $P(\text{k-th jump} \leq t < \text{(k+1)-th jump}) > 0 \quad \forall t > 0$

Long run behavior of continuous time MC

Remarks: Continuous time MCs are "aperiodic"

All irreducible continuous time MCs are "regular"

Example.



Thm. If $(X_t)_{t\geq 0}$ is irreducible, then there exists $\pi_0, \dots, \pi_N$

1) $\pi_i > 0$, $\sum_{i=0}^{N} \pi_i = 1$

2) $\lim_{t\to\infty} P_{ij}(t) = \pi_j$ for all i

3) $\pi = (\pi_0, \dots, \pi_N)$ is uniquely determined by

π is called limiting/stationary/equilibrium distribution of (X_t)

Long run behavior of continuous time MC

Remark about 3) : $\pi Q = 0$ is equivalent to $\forall t$

($\Rightarrow$) If $\pi Q = 0$, then using Kolmogorov backward equation

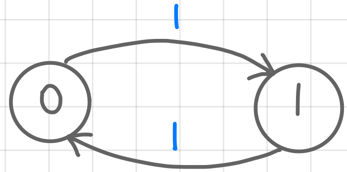
$$(\pi P(t))' =$$

so $\pi P(t)$ is independent of t. Since $P(0) = I$, we get

$$\forall t \qquad \pi P(t) =$$

($\Leftarrow$) If $\pi P(t) = \pi$, then $(\pi P(t))' = 0$. Using Kolmogorov forward equation

Example : Two-state MC



From Lecture 9: if $Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$, then

$$P(t) = I + \frac{1}{\alpha+\beta} Q - \frac{1}{\alpha+\beta} e^{-(\alpha+\beta)t} Q$$

. If $\alpha = \beta = 1$

$$\lim_{t \to \infty} P(t) =$$

Note, that the jump process (Y_n) does not have limiting distribution! $\tilde{P}^{Y_n} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Long run behavior of discrete time MC. Summary (2)

Let $(X_n)_{n\geq 0}$ be a disrete time MC on $\{0,1,\dots\}$ with stationary transition probability matrix $P=(P_{ij})_{i,j=0}^{\infty}$

Define $R_i=\min\{n: X_n=i\}$, $m_i=E(R_i \mid X_0=i)$ mean duration between visits

Thm. If $(X_n)_{n\geq 0}$ is recurrent irreducible aperiodic, then

$$\lim_{n\to\infty} P_{ij}^{(n)} = \frac{1}{m_j} \quad \forall j$$

If $\lim_{n\to\infty} P_{ij}^{(n)} > 0$ for some (all) j, then MC is positive recurrent

$\lim_{n\to\infty} P_{ij}^{(n)} = 0$ for some (all) j, then MC is null recurrent.

If (X_n) is positive recurrent, $(\pi_j)_{j=0}^{N}$, $\pi_j=\lim_{n\to\infty} P_{ij}^{(N)}$ is called stationary distribution, uniquely determined by

$$\pi_j=\sum_{i=0}^{\infty}\pi_i P_{ij} \quad \forall j, \quad \sum_{i=0}^{\infty}\pi_i=1, \quad \pi_i>0$$

Long run behavior of continuous time MC (2)

Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,\dots\}$

and let $(Y_n)_{n\geq 0}$ be the embedded jump chain.

Define $R_i = \min\{t > S_0 : X_t = i\}$,

$m_i = E(R_i \mid X_0 = i)$ – mean return time from i to i

If $m_i < \infty$, then i is positive recurrent (class property).

Thm 1) If $(X_t)_{t\geq 0}$ is irreducible, then

$$\lim_{t\to\infty} P_{ij}(t) =$$

2) $(X_t)_{t\geq 0}$ is positive recurrent iff there exists a (unique) solution $(\pi_j')_{j=0}^{\infty}$ to

in which case $\pi_j = \pi_j'$ and $(\pi_j)_{j=0}^{\infty}$ is called limiting/stationary distribution.

Remarks

1) Until now we discussed only the transition probabilities. But in order to describe completely MC (X_t) we need also the initial / starting distribution

$$\nu = (\nu_0, \nu_1, \dots), \quad \nu_i = P(X_0 = i)$$

$$(X_t) \longleftrightarrow (\nu, Q)$$

2) Distribution of X_{t_1} is given by $\nu P(t_1)$

$$P(X_{t_1} = i) =$$

More generally

$$P(X_0 = i_0, X_{t_1} = i_1, \dots, X_{t_n} = i_n) =$$

3) Stationary distribution remains unchaged in time

$$\pi P(t) = \pi \Rightarrow$$

Remarks

4) Similarly as in the discrete case, π_j gives the fraction of time spent in state j in long run

(compare with $\lim_{m \to \infty} E\left[\frac{1}{m} \sum_{n=0}^{m-1} \mathbb{1}_{\{X_n = j\}} \mid X_0 = i \right] = \pi_j$ for discrete time MC)

5) If we can find $(\pi_i)_{i=0}^{\infty}$ such that

then $(\pi_i)_{i=0}^{\infty}$ satisfies $\pi Q = 0$

Indeed, $\sum_{j=0}^{\infty} \pi_i q_{ij} = \pi_i \sum_{j=0}^{\infty} q_{ij} = 0 = \sum_{j=0}^{\infty} \pi_j q_{ji} = (\pi Q)_i$

Example: Birth and death processes

If we consider the birth and death process, the equation $\pi Q = 0$

takes the following form

where $\theta_i = \frac{\lambda_{i-1}}{\mu_i} \cdot \frac{\lambda_{i-2}}{\mu_{i-1}} \cdot \ldots \cdot \frac{\lambda_0}{\mu_1}$, $\theta_0 = 1$.

Then, $\sum_{i=0}^{\infty} \pi_i = 1$ implies that

If $\sum_{i=0}^{\infty} \theta_i < \infty$, then (X_t) is positive recurrent and $\pi_j =$

If $\sum_{i=0}^{\infty} \theta_i = \infty$, then $\pi_j = 0$ $\forall j$.

Example. Linear growth with immigration

Birth and death process, $\lambda_j = \lambda j + a$, $\mu_j = \mu j$ $(*)$

Using Kolmogorov's equations we showed (lecture 9)

that $E(X_t) \to \frac{a}{\mu - \lambda}$, $t \to \infty$, if $\mu > \lambda$.

What is the limiting distribution of X_t?

From the previous slide, $\pi_j = \frac{\theta_j}{\sum_{i=0}^{\infty} \theta_i}$, $\theta_j = \frac{\lambda_{j-1} \cdots \lambda_0}{\mu_j \cdots \mu_1}$

If we replace λ_j, μ_j by $(*)$, we get

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \left(1 - \frac{\lambda}{\mu}\right)^{a/\lambda} \frac{\frac{a}{\lambda}\left(\frac{a}{\lambda}+1\right)\cdots\left(\frac{a}{\lambda}+j-1\right)}{j!}, \quad j \geq 1$$

$$\pi_0 = \left(1 - \frac{\lambda}{\mu}\right)^{-\frac{a}{\lambda}}$$

What you should know for midterm 1 (minimum):

- definition of continuous time MC, Markov property, transition probabilities, generator
- representations of MC: infinitesimal (generator), jump-and-hold, transition probabilities, rate diagram and relations between them (in particular Q and $P(t)$)
- computing absorption probabilities and mean time to absorption
- computing stationary distributions for finite and infinite state MCs and interpretation of $(\pi_i)_{i=0}^{\infty}$
- basic properties of birth and death processes