MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: MC review. Conditioning on continuous random variables

Next: PK 7.1, Durrett 3.1

Week 4:

- HW3 due Saturday, April 29 on Gradescope
- Midterm 1: Friday, April 28

Remarks

4) Similarly as in the discrete case, π_j gives the fraction of time spent in state j in long run

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \mathbb{1}_{\{X_t=j\}}\,dt \,\middle|\, X_0=i\right] = \pi_j$$

(compare with $\lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m}\sum_{n=0}^{m-1} \mathbb{1}_{\{X_n=j\}} \,\middle|\, X_0=i\right] = \pi_j$ for discrete time MC)

5) If we can find $(\pi_i)_{i=0}^{\infty}$ such that $\pi_i q_{ij} = \pi_j q_{ji}$ then $(\pi_i)_{i=0}^{\infty}$ satisfies $\pi Q = 0$

Indeed, $\sum_{j=0}^{\infty} \pi_i q_{ij} = \pi_i \sum_{j=0}^{\infty} q_{ij} = 0 = \sum_{j=0}^{\infty} \pi_j q_{ji} = (\pi Q)_i$

Example: Birth and death processes

If we consider the birth and death process, the equation $\pi Q = 0$ takes the following form

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \mu_2 & -(\lambda_2+\mu_2) & \\ & & \ddots & \ddots \end{pmatrix}$$

$$-\lambda_0 \pi_0 + \mu_1 \pi_1 = 0$$
$$\lambda_0 \pi_0 - (\lambda_1+\mu_1)\pi_1 + \mu_2 \pi_2 = 0$$
$$\vdots$$
$$\lambda_{i-1}\pi_{i-1} - (\lambda_i+\mu_i)\pi_i + \mu_{i+1}\pi_{i+1} = 0$$

$$\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$$
$$\pi_2 = \frac{\lambda_1}{\mu_2}\pi_1 = \frac{\lambda_1\lambda_0}{\mu_2\mu_1}\pi_0$$
$$\pi_{i+1} = \frac{\lambda_i}{\mu_{i+1}}\pi_i = \cdots = \underbrace{\frac{\lambda_i \cdots \lambda_0}{\mu_{i+1}\cdots\mu_1}}_{\theta_{i+1}}\pi_0$$

where $\theta_i = \frac{\lambda_{i-1}}{\mu_i}\cdot\frac{\lambda_{i-2}}{\mu_{i-1}}\cdot \cdots \cdot\frac{\lambda_0}{\mu_1}$, $\theta_0 = 1$.

$$\pi_{i+1} = \theta_{i+1}\cdot\pi_0$$

Then, $\sum_{i=0}^{\infty}\pi_i = 1$ implies that $\pi_0 \sum_{i=0}^{\infty}\theta_i = 1$

If $\sum_{i=0}^{\infty}\theta_i < \infty$, then (X_t) is positive recurrent and $\pi_j = \frac{\theta_j}{\sum_{i=0}^{\infty}\theta_i}$

If $\sum_{i=0}^{\infty}\theta_i = \infty$, then $\pi_j = 0 \ \forall j$.

Example. Linear growth with immigration

Birth and death process, $\lambda_j = \lambda j + a$, $\mu_j = \mu j$ $(*)$

Using Kolmogorov's equations we showed (lecture 9)

that $E(X_t) \to \frac{a}{\mu-\lambda}$, $t \to \infty$, if $\mu > \lambda$.

What is the limiting distribution of X_t?

From the previous slide, $\pi_j = \frac{\theta_j}{\sum_{i=0}^{\infty} \theta_i}$, $\theta_j = \frac{\lambda_{j-1} \cdots \lambda_0}{\mu_j \cdots \mu_1}$

If we replace λ_j, μ_j by $(*)$, we get

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \left(1 - \frac{\lambda}{\mu}\right)^{a/\lambda} \frac{\frac{a}{\lambda}\left(\frac{a}{\lambda}+1\right) \cdots \left(\frac{a}{\lambda}+j-1\right)}{j!}, \quad j > 1$$

$$\pi_0 = \left(1 - \frac{\lambda}{\mu}\right)^{-\frac{a}{\lambda}}$$

What you should know for midterm 1 (minimum):

- definition of continuous time MC, Markov property, transition probabilities, generator
- representations of MC: infinitesimal (generator), jump-and-hold, transition probabilities, rate diagram and relations between them (in particular Q and $P(t)$)
- computing absorption probabilities and mean time to absorption
- computing stationary distributions for finite and infinite state MCs and interpretation of $(\pi_i)_{i=0}^{\infty}$
- basic properties of birth and death processes

Conditioning on continuous r.v.

<u>Def.</u> Let X and Y be jointly continuous random variables with joint probability density function $f_{X,Y}(x,y)$. We call the function

$$f_{X|Y}(x|y) := \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{for } y \text{ s.t. } f_Y(y)>0$$

the conditional probability density function of X given $Y=y$.

The function $F_{X|Y}(x|y) := \int_{-\infty}^{x} f_{X|Y}(s|y)\, ds$

is called conditional CDF of X given $Y=y$

Conditional expectation

Def. Let X and Y be jointly continuous random variables, let $f_{X|Y}(x|y)$ be a conditional distribution of X given $Y=y$ and let $g: \mathbb{R} \to \mathbb{R}$ be a function for which $E(|g(X)|) < \infty$.

Then we call

$$E(g(X) \mid Y=y) := \int_{-\infty}^{+\infty} g(x) f_{X|Y}(x|y)\, dx \quad \text{if } f_Y(y) > 0$$

the conditional expectation of $g(X)$ given $Y=y$.

In particular, if $g(x) = \mathbb{1}_A(x)$ indicator of set A, then

$$E(\mathbb{1}_A(X) \mid Y=y) = \int_A f_{X|Y}(x|y)\, dx = P(X \in A \mid Y=y)$$

Remark

If Y is a continuous random variable, then

$$P(Y=y)=0 \quad \text{for all } y \in \mathbb{R}$$

Therefore, we cannot define $P(X \in A \mid Y=y)$ as

$$P(X \in A \mid Y=y) = \cancel{\frac{P(X \in A, Y=y)}{P(Y=y)}}$$

On the other hand consider example:

X, Y i.i.d. $X, Y \sim \text{Unif}[0,1]$, $Z = X - Y$

If $Y = \frac{1}{2}$, $Z = X - \frac{1}{2} \sim \text{Unif}[-\frac{1}{2}, \frac{1}{2}]$

makes sense

Intuitive explanation / derivation

$$P(X \in [x, x+\Delta x], Y \in [y, y+\Delta y])$$
$$= f_{X,Y}(x,y) \cdot \Delta x \cdot \Delta y + o(\Delta x \Delta y) \quad \text{as } \begin{array}{l} \Delta x \to 0 \\ \Delta y \to 0 \end{array}$$

Using the multiplication rule $(f_Y(y) > 0 \text{ on } [y, y+\Delta y])$

$$P(X \in [x, x+\Delta x], Y \in [y, y+\Delta y])$$
$$= P(X \in [x, x+\Delta x] \mid Y \in [y, y+\Delta y]) \, P(Y \in [y, y+\Delta y])$$

$$\frac{P(X \in [x, x+\Delta x] \mid Y \in [y, y+\Delta y])}{\Delta x} = \frac{\dfrac{P(X \in [x, x+\Delta x], Y \in [y, y+\Delta y])}{\Delta x \, \Delta y}}{\dfrac{P(Y \in [y, y+\Delta y])}{\Delta y}}$$

$$\Big\downarrow \; \Delta x \to 0 \qquad\qquad \Big\downarrow \; \begin{array}{l} \Delta x \to 0 \\ \Delta y \to 0 \end{array}$$

$$\text{"} f_X(x \mid Y \in [y, y+\Delta y]) \text{"} \qquad\qquad \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$\Big\downarrow \; \Delta y \to 0$$

$$\text{"} f_{X|Y}(x \mid y) \text{"}$$