MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Conditioning on continuous random variables

Next: PK 7.1, Durrett 3.1

Week 4:

- HW3 due Saturday, April 29 on Gradescope
- Midterm 1: Friday, April 28

Properties of conditional probability/expectation

1) $P(a < X < b, c < Y < d) =$

2) $P(a < X < b) =$

3) $E(g(X)) =$

Further properties of conditional expectation (PK, p.50)

4) $E(c_1 g_1(X_1) + c_2 g_2(X_2) \mid Y=y) = c_1 E(g_1(X_1) \mid Y=y) + c_2 E(g_2(X_2) \mid Y=y)$

5) $E(v(X,Y) \mid Y=y) =$

In particular, $E(v(X,Y)) =$

6) $E(g(X)h(Y)) = \int_{-\infty}^{+\infty} h(y) E(g(X) \mid Y=y) f_Y(y)\, dy$

$$= E(h(Y) E(g(X) \mid Y))$$

7) $E(g(X) \mid Y=y) = E(g(X))$ if X and Y are independent

Example 1

Let (X, Y) be jointly continuous r.v.s with density

$$f_{X,Y}(x,y) = \frac{1}{y} e^{-\frac{x}{y} - y}, \quad x, y > 0$$

Compute the conditional density of X given $Y=y$.

1) Compute the marginal density of Y

2) Compute the conditional density

Example 1 (cont.)

Suppose that $Y \sim Exp(1)$, and X has exponential distribution with parameter $\frac{1}{Y}$. Compute $E(X)$

$$E(X) = \int_0^\infty \int_0^\infty x \frac{1}{y} e^{-\frac{x}{y} - y} \, dx \, dy$$

Example 2 : continuous and discrete r.v.s

Let $N \in \mathbb{N}$, $P \sim \mathcal{U}nif[0,1]$, $X \sim Bin(N, P)$

What is the distribution of X?

$$P(X=k) =$$

$$=$$

$$=$$

$$=$$

Example 3

Let X and Y be i.i.d. $\mathrm{Exp}(\lambda)$ r.v.

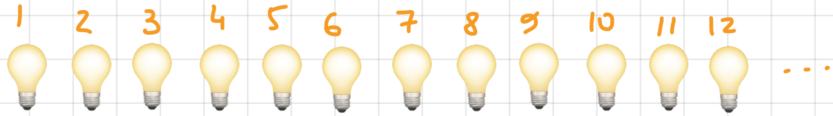
Define $Z = \frac{X}{Y}$. Compute the density of Z.

- If $X \sim \mathrm{Exp}(\lambda)$, then for $\alpha > 0$ $\alpha X \sim \mathrm{Exp}\left(\frac{\lambda}{\alpha}\right)$

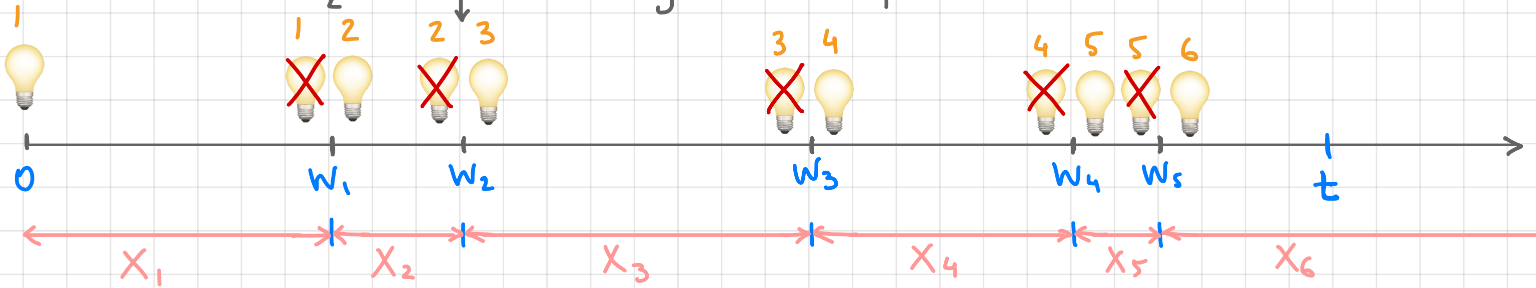
- .

Renewal process

Imagine lightbulbs



"renewal" = lightbulb replacement



X_i – lifetime of the lightbulb #i, W_i = time of i-th "renewal"

Lightbulbs are identical $\Rightarrow$ X_i are i.i.d.

Let $N(t)$ denote the number of renewals up to time t

- What are the properties of $(N(t))_{t\geq 0}$?
- How they depend on the distribution of X_i?

Renewal process. Definition

Def. Let $\{X_i\}_{i\ge 1}$ be i.i.d. r.v.s, $X_i > 0$.

Denote $W_n := X_1 + \dots + X_n$, $n \ge 1$, and $W_0 := 0$.

We call the counting process

the renewal process.

Remarks. 1) W_n are called the waiting/renewall times

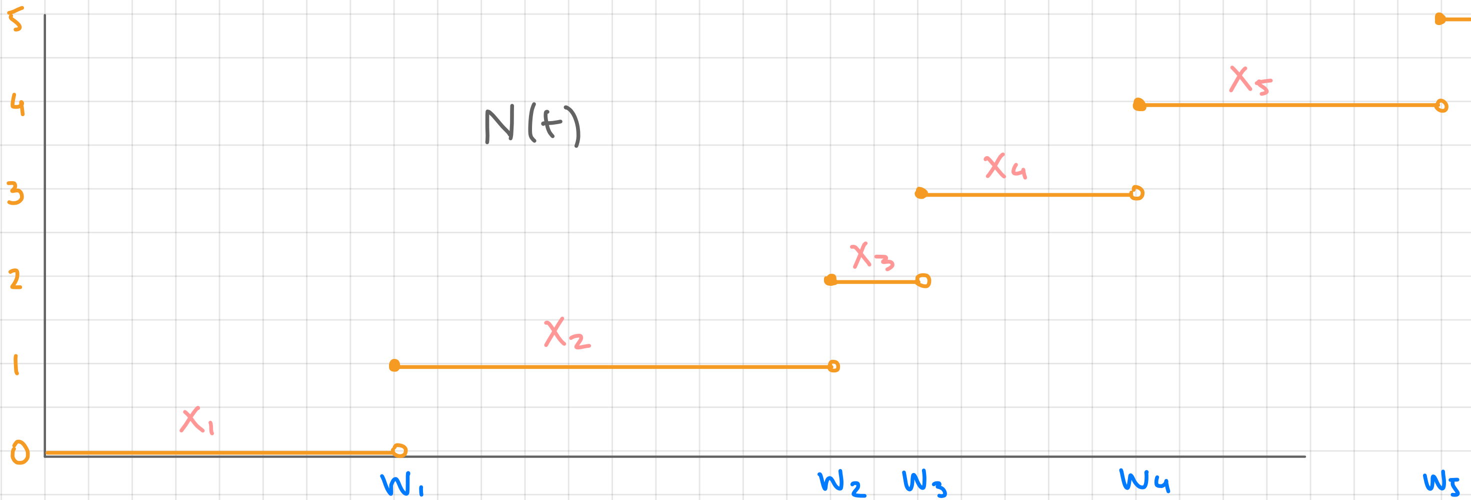
X_i are called the interrenewal times

2) $N(t)$ is characterised by the distribution of $X_i > 0$

3) More generally, we can define for $0 \le a < b < \infty$

$$N((a,b]) = \#\{k : a < W_k \le b\}$$

Renewal process. Definition



Remarks 1) (*) implies that $(N(t))_{t\geq 0}$ is determined by $(W_k)_{k\geq 0}$, so sometimes $(W_k)_{k\geq 0}$ is called renewal process

2) For any t, $W_{N(t)} \leq t < W_{N(t)+1}$

Convolutions of c.d.f.s

Suppose that X and Y are independent r.v.s

$F: \mathbb{R} \to [0,1]$ is the c.d.f. of X (i.e. $P(X \le t) = F(t)$).

$G: \mathbb{R} \to [0,1]$ is the c.d.f. of Y

- if Y is discrete, then

$$F_{X+Y}(t) = P(X+Y \le t) =$$

$$=$$

$$=$$

- if Y is continuous, then

$$F_{X+Y}(t) = P(X+Y \le t) =$$

Distribution of W_k

Let $X_1, X_2, \ldots$ be i.i.d. r.v.s, $X_i > 0$, and let $F: \mathbb{R} \to [0,1]$ be the c.d.f. of X_i (we call F the interoccurrence or interrenewal distribution). Then

- $F_1(t) := F_{W_1}(t) = P(W_1 \leq t) = P(X_1 \leq t) = F(t)$
- $F_2(t) := F_{W_2}(t) = F_{X_1+X_2}(t) =$
- $F_3(t) := F_{W_3}(t) =$
- More generally,

$$F_n(t) := F_{W_n}(t) = P(W_n \leq t) =$$

Remark: $F^{*(n+1)}(t) = \int_0^t F^{*n}(t-x)\,dF(x) = \int_0^t F(t-x)\,dF^{*n}(x)$

Renewal function

<u>Def</u>. Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. We call

the renewal function.

<u>Proposition</u> 1. $M(t) =$

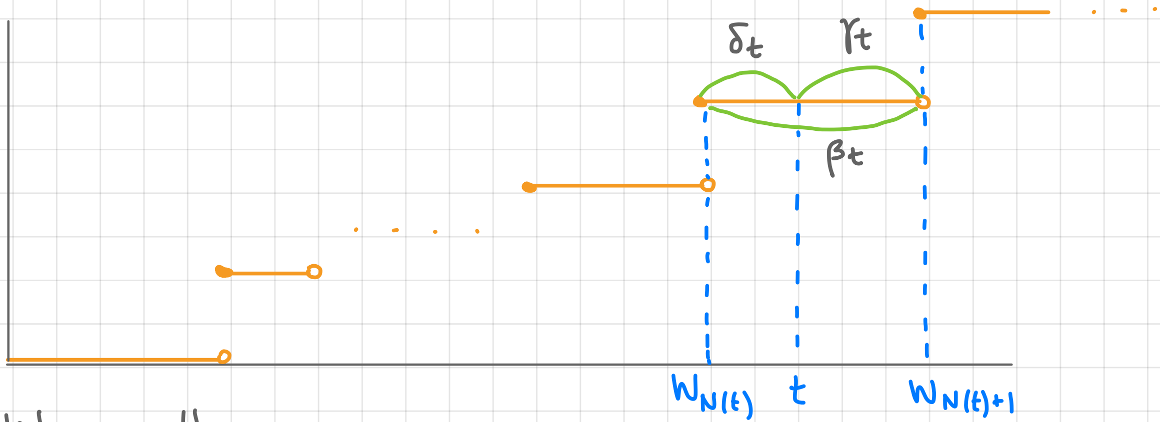
<u>Proof</u>. $M(t) = E(N(t)) =$

$=$

$=$

Related quantities

Let $N(t)$ be a renewal process.



Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks 1)

2)

Expectation of W_n

Proposition 2. Let $N(t)$ be a renewal process with interrenewal times $X_1, X_2, \dots$ and renewal times $(W_n)_{n \geq 1}$. Then

$$E(W_{N(t)+1}) = E(X_1)\ E(N(t)+1)$$
$$= \mu(M(t)+1)$$

where $\mu = E(X_1)$.

Proof. $E(W_{N(t)+1}) =$

$E(X_2 + \dots + X_{N(t)+1}) =$

$=$

Expectation of W_n

$$E\left(\sum_{j=2}^{N(t)+1} X_j\right) =$$

$$=$$

Since $N(t) \geq j-1 \iff W_{j-1} \leq t \iff X_1 + X_2 + \dots + X_{j-1} \leq t$

$$=$$

$$=$$

$$=$$

Remark For proof in PK take $1 = \sum_{i=1}^{\infty} \mathbb{1}_{\{N(t)=i\}}$.

Renewal equation

Proposition 3. Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. Then $M(t) = E(N(t))$ satisfies

Proof. We showed in Proposition 1 that

$$M = \sum_{n=1}^{\infty} F^{*n}.$$

Then $\quad M * F =$