MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Renewal processes

Next: PK 7.2-7.3, Durrett 3.1

Week 4:

- HW4 due Friday, May 5 on Gradescope

Renewal process. Definition

Def. Let $\{X_i\}_{i\geq 1}$ be i.i.d. r.v.s, $X_i > 0$.

Denote $W_n := X_1 + \dots + X_n$, $n \geq 1$, and $W_0 := 0$.

We call the counting process

$$N(t) = \#\{k>0 : W_k \leq t\} = \max\{n : W_n \leq t\}$$

the renewal process.

Remarks. 1) W_n are called the waiting/renewall times

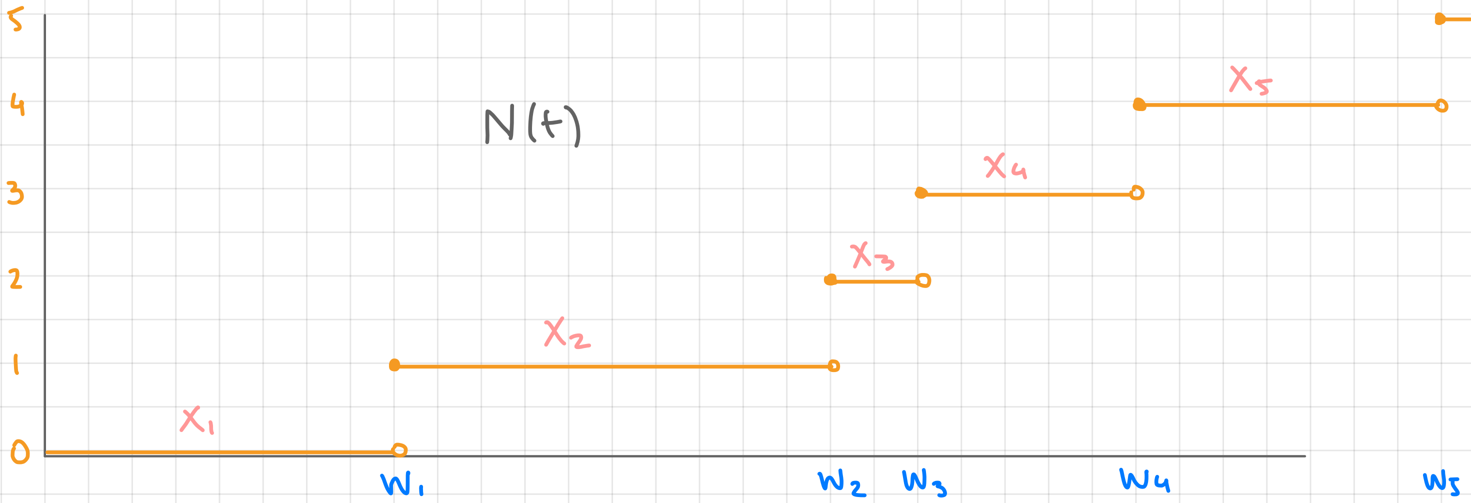
X_i are called the interrenewal times

2) $N(t)$ is characterised by the distribution of $X_i > 0$

3) More generally, we can define for $0 \leq a < b < \infty$

$$N((a,b]) = \#\{k : a < W_k \leq b\}$$

Renewal process. Definition



Remarks 1) (*) implies that $(N(t))_{t\geq 0}$ is determined by $(W_k)_{k\geq 0}$, so sometimes $(W_k)_{k\geq 0}$ is called renewal process

2) For any t, $W_{N(t)} \leq t < W_{N(t)+1}$

Convolutions of c.d.f.s

Suppose that X and Y are independent r.v.s

$F: \mathbb{R} \to [0,1]$ is the c.d.f. of X (i.e. $P(X \leq t) = F(t)$).

$G: \mathbb{R} \to [0,1]$ is the c.d.f. of Y

- if Y is discrete, then

$$F_{X+Y}(t) = P(X+Y \leq t) =$$

$$=$$

$$=$$

- if Y is continuous, then

$$F_{X+Y}(t) = P(X+Y \leq t) =$$

Distribution of W_k

Let $X_1, X_2, \dots$ be i.i.d. r.v.s, $X_i > 0$, and let $F : \mathbb{R} \to [0,1]$ be the c.d.f. of X_i (we call F the interoccurrence or interrenewal distribution). Then

- $F_1(t) := F_{W_1}(t) = P(W_1 \le t) = P(X_1 \le t) = F(t)$
- $F_2(t) := F_{W_2}(t) = F_{X_1+X_2}(t) =$
- $F_3(t) := F_{W_3}(t) =$
- More generally,

$$F_n(t) := F_{W_n}(t) = P(W_n \le t) =$$

Remark: $F^{*(n+1)}(t) = \int_0^t F^{*n}(t-x)\, dF(x) = \int_0^t F(t-x)\, dF^{*n}(x)$

Renewal function

<u>Def</u>. Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. We call

the renewal function.

<u>Proposition</u> 1. $M(t) =$

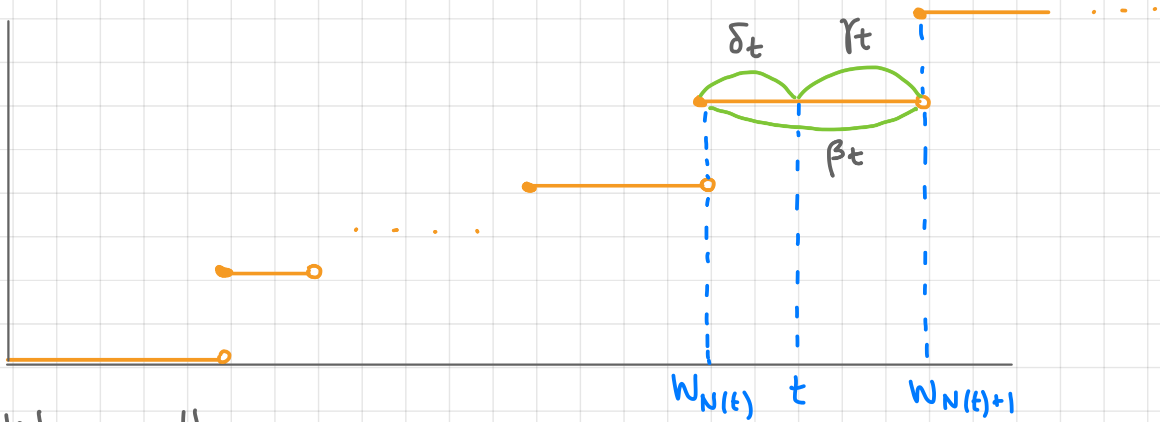
<u>Proof</u>. $M(t) = E(N(t)) =$

$=$

$=$

Related quantities

Let $N(t)$ be a renewal process.



Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks 1)

2)

Expectation of W_n

Proposition 2. Let $N(t)$ be a renewal process with interrenewal times $X_1, X_2, \dots$ and renewal times $(W_n)_{n\geq 1}$. Then

$$E(W_{N(t)+1}) = E(X_1)\, E(N(t)+1)$$
$$= \mu(M(t)+1)$$

where $\mu = E(X_1)$.

Proof. $E(W_{N(t)+1}) =$

$E(X_2 + \cdots + X_{N(t)+1}) =$

$=$

Expectation of W_n

$$E\left(\sum_{j=2}^{N(t)+1} X_j\right) =$$

$$=$$

Since $\quad N(t) \geq j-1 \iff W_{j-1} \leq t \iff X_1 + X_2 + \cdots + X_{j-1} \leq t$

$$=$$

$$=$$

$$=$$

Remark For proof in PK take $1 = \sum_{i=1}^{\infty} \mathbb{1}_{\{N(t)=i\}}$.

Renewal equation

<u>Proposition 3</u>. Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. Then $M(t) = E(N(t))$ satisfies

<u>Proof</u>. We showed in Proposition 1 that

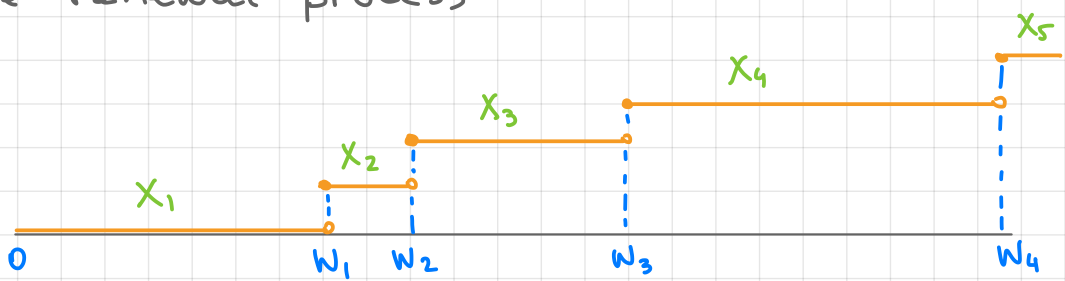
$$M = \sum_{n=1}^{\infty} F^{*n}.$$

Then $M * F =$

Poisson process as a renewal process

The Poisson process $N(t)$ with rate $\lambda > 0$ is a renewal process with $F(x) = 1 - e^{-\lambda x}$.

- sojourn times S_i are i.i.d., $S_i \sim Exp(\lambda)$
- S_i represent intervals between two consecutive events (arrivals of customers)
- $W_n = \sum_{i=0}^{n-1} S_i$
- we can take $X_i = S_{i-1}$ in the definition of the renewal process



Poisson process as a renewal process

We know that $N(t) \sim \text{Pois}(\lambda t)$, so in particular

$$E(N(t)) = \lambda t$$

Example Compute $M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$ for PP

$F_2(t) =$

Denote $\varphi_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$:

$$\varphi_k * F(t) =$$

$F * F(t) =$

$F^{*3}(t) =$

$\vdots$

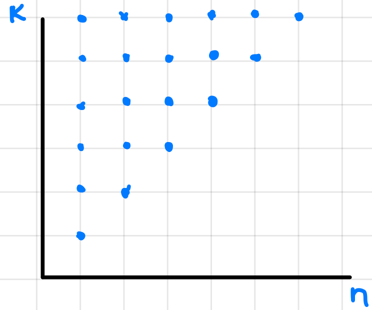
$F^{*n}(t) =$

Poisson process as a renewal process (cont.)

$$\sum_{n=1}^{\infty} F^{*n}(t) =$$

$$=$$

$$=$$



$$M(t) =$$