MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Poisson process as a renewal process

Next: PK 7.2-7.3, Durrett 3.1

Week 5:

- HW4 due Friday, May 5 on Gradescope

Expectation of W_n

Proposition 2. Let $N(t)$ be a renewal process with interrenewal times $X_1, X_2, \dots$ and renewal times $(W_n)_{n\geq 1}$. Then

$$E(W_{N(t)+1}) = E(X_1)\, E(N(t)+1)$$
$$= \mu (M(t)+1)$$

where $\mu = E(X_1)$.

Proof. $E(W_{N(t)+1}) = E(X_1 + X_2 + \cdots + X_{N(t)+1}) = \overset{\mu}{\overset{\|}{E(X_1)}} + E(X_2 + \cdots + X_{N(t)+1})$

$$E(X_2 + \cdots + X_{N(t)+1}) = E(X_2 \mid N(t)=1)P(N(t)=1)$$
$$+ E(X_2 + X_3 \mid N(t)=2)P(N(t)=2)$$
$$+ E(X_2 + X_3 + X_4 \mid N(t)=3)P(N(t)=3)$$
$$\vdots$$
$$+ E\left(\sum_{k=2}^{n+1} X_k \mid N(t)=n\right) P(N(t)=n) + \cdots$$

$$= \sum_{n=1}^{\infty} E(X_2 \mid N(t)=n)P(N(t)=n) + \sum_{n=2}^{\infty} E(X_3 \mid N(t)=n)P(N(t)=n) + \cdots$$

Renewal equation

Proposition 3. Let $(N(t))_{t\geq 0}$ be a renewal process with interrenewal distribution F. Then $M(t) = E(N(t))$ satisfies

Proof. We showed in Proposition 1 that

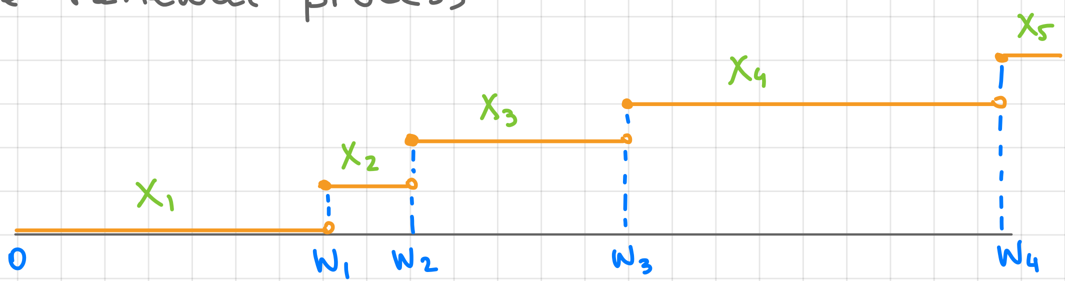
$$M = \sum_{n=1}^{\infty} F^{*n}.$$

Then $\quad M * F =$

Poisson process as a renewal process

The Poisson process $N(t)$ with rate $\lambda > 0$ is a renewal process with $F(x) = 1 - e^{-\lambda x}$.

- sojourn times S_i are i.i.d., $S_i \sim Exp(\lambda)$
- S_i represent intervals between two consecutive events (arrivals of customers)
- $W_n = \sum_{i=0}^{n-1} S_i$
- we can take $X_i = S_{i-1}$ in the definition of the renewal process



Poisson process as a renewal process

We know that $N(t) \sim \text{Pois}(\lambda t)$, so in particular

$$E(N(t)) = \lambda t$$

Example Compute $M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$ for PP

$F_2(t) =$

Denote $\varphi_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$:

$$\varphi_k * F(t) =$$

$F * F(t) =$

$F^{*3}(t) =$

$\vdots$

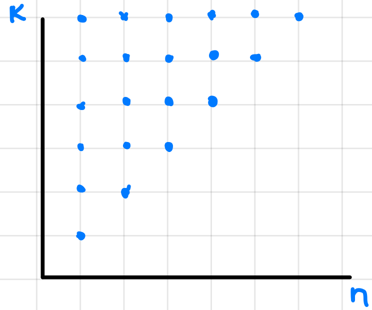
$F^{*n}(t) =$

Poisson process as a renewal process (cont.)

$$\sum_{n=1}^{\infty} F^{*n}(t) =$$

$$=$$

$$=$$



$$M(t) =$$

Renewal density

Proposition Let $N(t)$ be a renewal process with continuous interrenewal times X_i having density $f(x)$. Denote

$m(t) = \sum_{n=1}^{\infty} f^{*n}(t)$. Then

and (*)

renewal density

Proof: $\frac{d}{dt} F^{*n}(t) =$ $\blacksquare$

Example: Compute the renewal density for PP using (*).

$f(x) = \lambda e^{-\lambda x}$, so (*) becomes

$m(t) =$

$=$

(cont.)

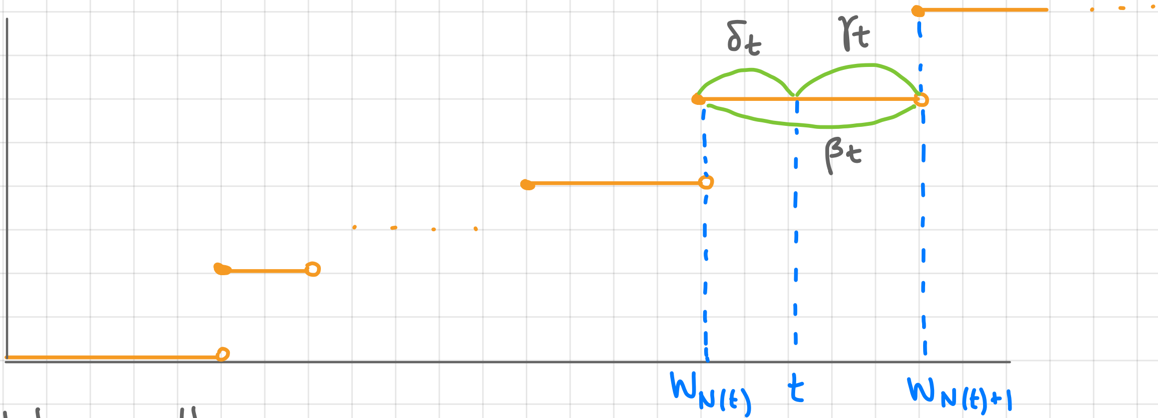
$e^{\lambda t} m(t) =$ $\leftarrow$ differentiate

$\Rightarrow$

Indeed,

Excess life and current life of PP (summary)

Recall: Let $N(t)$ be a renewal process.



Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks 1) $\gamma_t > h \geq 0$ iff $N(t+h) = N(t)$

2) $t \geq h$ and $\delta_t \geq h$ iff $N(t-h) = N(t)$

Excess life and current life of PP

Let $N(t)$ be a PP. Then

- excess life

$P(\gamma_t > x) =$

- current life δ_t

$$P(\delta_t > x) = \begin{cases} & \\ & \end{cases}$$

- total life $\beta_t = \gamma_t + \delta_t$

$E(\gamma_t + \delta_t) =$

$=$

Excess life and current life of PP (cont.)

- Joint distribution of (γ_t, δ_t)

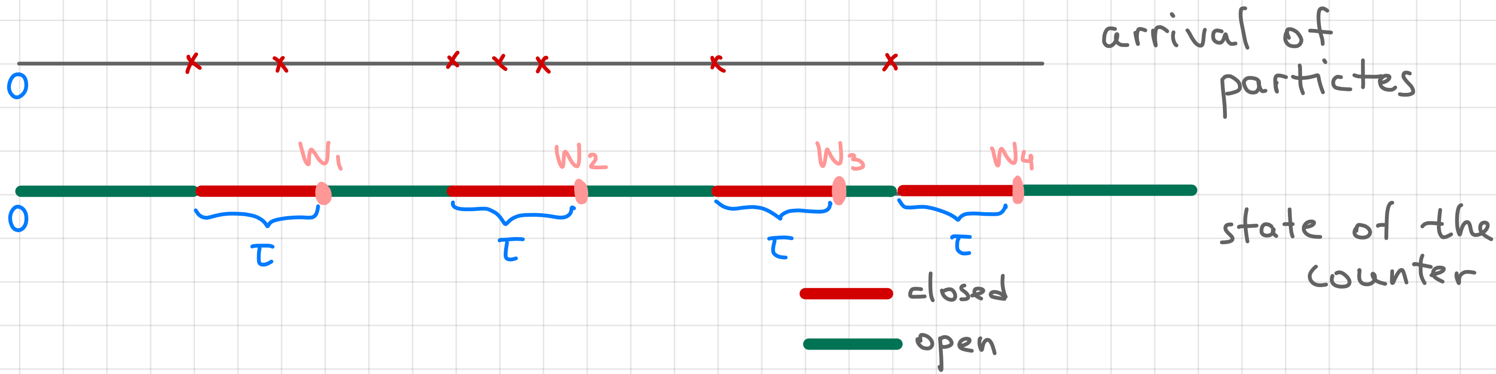
$$P(\gamma_t > x, \delta_t > y) = \begin{cases} & \\ & \end{cases}$$

$\Rightarrow$

Other renewal processes

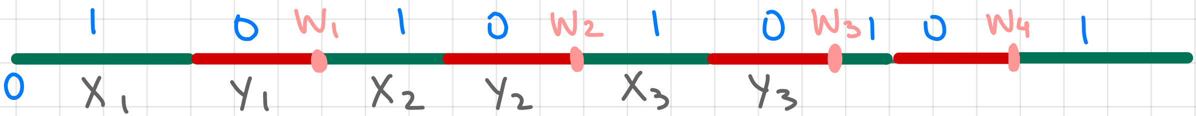
- traffic flow : distances between successive cars are assumed to be i.i.d. random variables

- counter process: particles/signals arrive on a device and lock it for time τ ; particles arrive according to a PP; times at which the counter unlocks form a renewal process



Other renewal processes

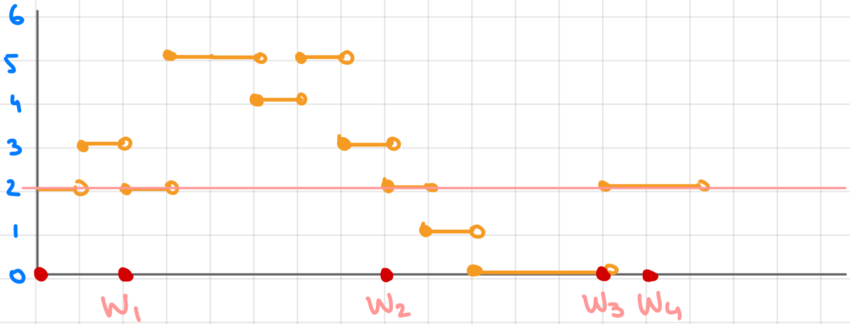
- more generally, if a component has two states (0/1, operating/non-operating etc), switches between them, times spent in 0 are X_i, times spent in 1 are Y_i, $(X_i)_{i=1}^{\infty}$ i.i.d., $(Y_i)_{i=1}^{\infty}$ i.i.d., then the times of switching from 0 to 1 form a renewal process with interrenewal times $X_i + Y_i$



Other renewal processes

- Markov chains: if $(Y_n)_{n\geq 0}$, $Y_n \in \{0,1,\dots\}$ is a recurrent MC starting from $Y_0 = k$, then the times of returns to state k form a renewal process. More precisely define

$$W_1 = \min\{n > 0 : Y_n = k\}$$
$$W_p = \min\{n > W_{p-1} : Y_n = k\}$$



Example with $k=2$

Similarly for continuous time MCs.

Strong Markov property!

Other renewal processes

- Queues. Consider a single-server queueing process

(i) if customer arrival times form a renewal process then the times of the starts of successive idle periods generate a second renewal time

(ii) if customes arrive according to a Poisson process, then the times when the server passes from busy to free form a renewal process

Asymptotic behavior

Asymptotic behavior of renewal processes

Let $N(t)$ be a renewal process with interrenewal times X_i, $X_i \in (0, \infty)$.

Thm.

Proof. $N(t)$ is nondecreasing, therefore N_∞ is the total number of events ever happened.

Thm (Pointwise renewal thm).

Elementary Renewal Theorem

Thm. If $M(t) = E(N(t))$ and $E(X_1) = \mu$, then

Proof. (Only for bounded X_i: $\exists\ K$ s.t. $P(X_1 \leq K) = 1$)

First note that

In lecture 11 we showed that ,

so $M(t) =$

$$\frac{M(t)}{t} =$$

If $X_i \leq K$, then

Asymptotic distribution of $N(t)$

Thm. Let $N(t)$ be a renewal process with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$, then

1)

2)

No proof.