

MATH180C: Introduction to Stochastic Processes II

<https://mathweb.ucsd.edu/~ynemish/teaching/180c>

Today: Poisson process as a renewal
process

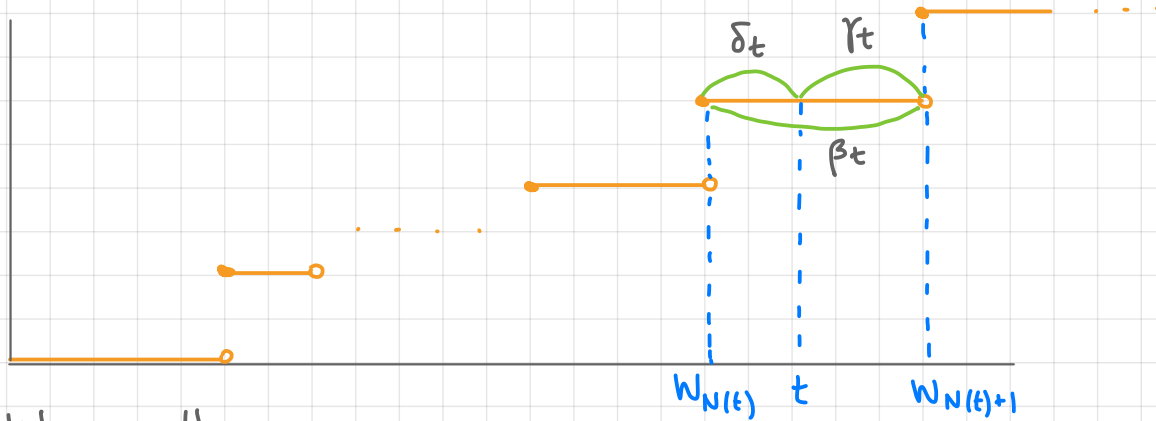
Next: PK 7.2-7.3, Durrett 3.1

Week 5:

- HW4 due Friday, May 5 on Gradescope

Excess life and current life of PP (summary)

Recall: Let $N(t)$ be a renewal process.



Def. We call

- $\gamma_t := W_{N(t)+1} - t$ the excess (or residual) lifetime
- $\delta_t := t - W_{N(t)}$ the current life (or age)
- $\beta_t := \gamma_t + \delta_t$ the total life

Remarks 1) $\gamma_t > h \geq 0$ iff $N(t+h) = N(t)$

2) $t \geq h$ and $\delta_t \geq h$ iff $N(t-h) = N(t)$

Excess life and current life of PP

Let $N(t)$ be a PP. Then

- excess life

$$P(\gamma_t > x) =$$

- current life δ_t

$$P(\delta_t > x) = \left\{ \right.$$

- total life $\beta_t = \gamma_t + \delta_t$

$$E(\gamma_t + \delta_t) =$$

=

Excess life and current life of PP (cont.)

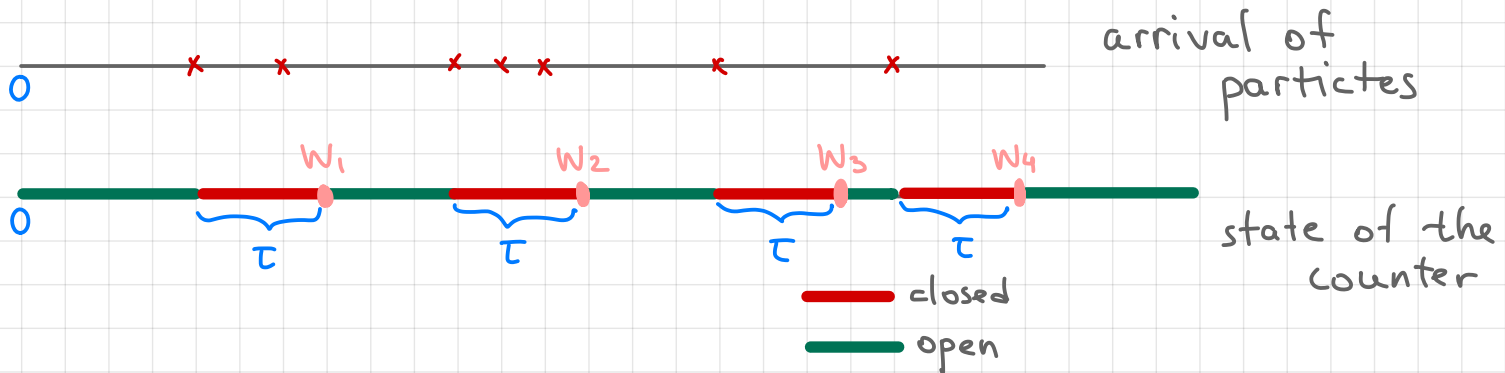
- Joint distribution of (γ_t, δ_t)

$$P(\gamma_t > x, \delta_t > y) = \left\{ \right.$$

\Rightarrow

Other renewal processes

- traffic flow : distances between successive cars are assumed to be i.i.d. random variables
- counter process: particles/signals arrive on a device and lock it for time τ ; particles arrive according to a PP; times at which the counter unlocks form a renewal process



Other renewal processes

- more generally, if a component has two states (0/1, operating/non-operating etc), switches between them, times spent in 0 are X_i , times spent in 1 are Y_i , $(X_i)_{i=1}^{\infty}$ i.i.d., $(Y_i)_{i=1}^{\infty}$ i.i.d., then the times of switching from 0 to 1 form a renewal process with interrenewal times $X_i + Y_i$

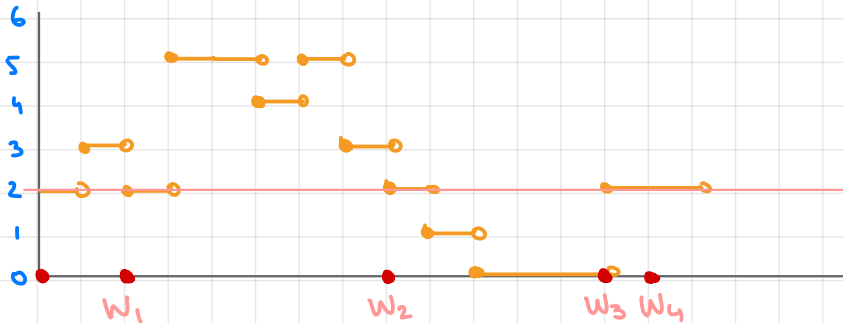


Other renewal processes

- Markov chains: if $(Y_n)_{n \geq 0}$, $Y_n \in \{0, 1, \dots\}$ is a recurrent MC starting from $Y_0 = k$, then the times of returns to state k form a renewal process. More precisely

$$\text{define } W_1 = \min \{n > 0 : Y_n = k\}$$

$$W_p = \min \{n > W_{p-1} : Y_n = k\}$$



Example with $k=2$

Similarly for continuous time MCs.

Strong Markov property!

Other renewal processes

- Queues. Consider a single-server queuing process



- (i) if customer arrival times form a renewal process then the times of the starts of successive idle periods generate a second renewal time
- (ii) if customers arrive according to a Poisson process, then the times when the server passes from busy to free form a renewal process

Asymptotic behavior

Asymptotic behavior of renewal processes

Let $N(t)$ be a renewal process with interrenewal times X_i , $X_i \in (0, \infty)$.

Thm.

Proof. $N(t)$ is nondecreasing, therefore

N_∞ is the total number of events ever happened.

Thm (Pointwise renewal thm).

Elementary Renewal Theorem

Thm. If $M(t) = E(N(t))$ and $E(X_1) = \mu$, then

Proof. (Only for bounded X_i : $\exists K$ s.t. $P(X_i \leq K) = 1$)

First note that

In lecture 11 we showed that

$$\text{so } M(t) =$$

$$\frac{M(t)}{t} =$$

If $X_i \leq K$, then

Asymptotic distribution of $N(t)$

Thm. Let $N(t)$ be a renewal process with
 $E(X_1) = \mu$ and $\text{Var}(X_1) = \sigma^2$, then

1)

2)

No proof.