MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 6:

- HW5 due Friday, May 12 on Gradescope

Asymptotic behavior of renewal processes

Let $N(t)$ be a renewal process with interrenewal times X_i, $X_i \in (0, \infty)$.

<u>Thm</u>.

$$P\left(\lim_{t\to\infty} N(t) = +\infty\right) = 1$$

<u>Proof</u>. $N(t)$ is nondecreasing, therefore $\exists \lim_{t\to\infty} N(t) =: \underset{\substack{\Downarrow \\ (0,+\infty)\cup\{+\infty\}}}{N_\infty}$

N_∞ is the total number of events ever happened.

$N_\infty \le k$ if and only if $W_{k+1} = \infty$

if and only if $X_i = \infty$ for some $1 \le i \le k+1$

$$P(N_\infty < \infty) = P(X_i = \infty \text{ for some } 1 \le i \le k+1) = P\left(\bigcup_{i=1}^{k+1} \{X_i = \infty\}\right) \le \sum_{i=1}^{k+1} P(X_i = \infty) = 0$$

<u>Thm</u> (Pointwise renewal thm).

Elementary Renewal Theorem

Thm. If $M(t) = E(N(t))$ and $E(X_1) = \mu$, then

Proof. (Only for bounded X_i: $\exists\, K$ s.t. $P(X_1 \le K) = 1$)

First note that

In lecture 11 we showed that

,

so $M(t) =$

$\frac{M(t)}{t} =$

If $X_i \le K$, then

Asymptotic distribution of $N(t)$

<u>Thm</u>. Let $N(t)$ be a renewal process with $E(X_1)=\mu$ and $Var(X_1)=\sigma^2$, then

1)

2)

No proof.

Elementary renewal theorem and continuous X_i's

Two more results (without proofs) about the limiting behaviour of $M(t)$ for models with continuous interrenewal times.

<u>Thm</u>. Let $E(X_1) = \mu$ and let $m(t) = \frac{d}{dt} M(t)$ be the renewal density. Then

<u>Remark</u> $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha$ does not imply in general $\lim_{t\to\infty} f'(t) = \alpha$

(E.g., take $f(t) = t + \sin(t)$)

<u>Thm</u>. If additionally $Var(X_1) = \sigma^2$, then

Example: $X_i \sim \text{Gamma}(2,1)$

Let $N(t)$ be a renewal process with interrenewal times X_i having Gamma distribution with parameters $(2,1)$ i.e., $f_{X_1}(t) = t e^{-t}$. Then from the properties of the Gamma distribution (or from direct computations)

$$X_1 + \cdots + X_n \sim \text{Gamma}(2n, 1), \text{ so}$$

$$f^{*n}(t) =$$

We can compute the renewal density

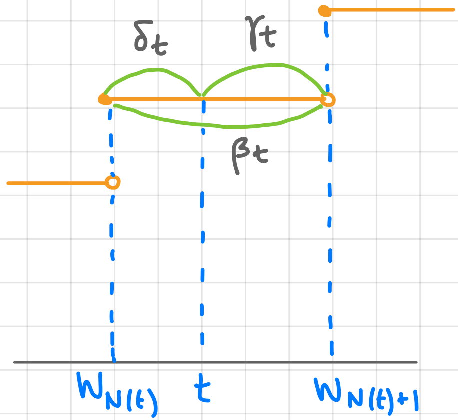
$$m(t) =$$

so that $M(t) =$

Finally, $E(X_1) = \mu =$, $Var(X_1) = \sigma^2 =$, so $\frac{\sigma^2 - \mu^2}{2\mu^2} =$

Joint distribution of age and excess life

From the definition of γ_t and δ_t



$P(\delta_t \geq x, \gamma_t > y)$ $\quad (x \leq t)$

$=$

- Partition wrt the values of $N(t)$

$=$

condition on the value of W_k (c.d.f. of W_k is $F^{*k}(t)$)

$=$

$=$

$=$

Limiting distribution of age and excess life

Assume that X_i are continuous. Then

$P(\delta_t \geq x, \gamma_t > y) =$

$=$

$=$

$=$

Recall that $\varepsilon(s) := m(s) - \frac{1}{\mu} \to 0$ as $s \to \infty$ $(\mu = E(X_1))$. Then

$\lim_{t \to \infty} P(\delta_t \geq x, \gamma_t > y) =$

$=$

Joint/limiting distribution of (γ_t, δ_t)

Thm. Let $F(t)$ be the c.d.f. of the interrenewal times. Then

(a) $$P(\gamma_t > y, \delta_t \geq x) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u))\, dF^{*k}(u)$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u))\, dM(u)$$

(b) if additionally the interrenewal times are continuous,

$$\lim_{t \to \infty} P(\gamma_t > y, \delta_t \geq x) = \frac{1}{\mu} \int_{x+y}^{\infty} (1 - F(w))\, dw \qquad (*)$$

If we denote by $(\gamma_\infty, \delta_\infty)$ a pair of r.v.s with distribution $(*)$ then γ_∞ and δ_∞ are continuous r.v.s with densities

$$f_{\gamma_\infty}(x) = f_{\delta_\infty}(x) =$$

Example

Renewal process (counting earthquakes in California) has interrenewal times uniformly distributed on $[0,1]$ (years).

(a) What is the long-run probability that an earthquake will hit California within 6 months?

(b) What is the long-run probability that it has been at most 6 months since the last earthquake?

Key renewal theorem

Suppose $H(t)$ is an unknown function that satisfies

$$H(t) = h(t) + H * F(t) \quad (*)$$

renewal equation

E.g.: $M(t) = F(t) + M * F(t)$,

$m(t) = f(t) + m * F(t) = f(t) + m * f(t)$

Remark about notation

- Convolution with c.d.f.: $g * F(t) = \int_{-\infty}^{+\infty} g(t-x)\, dF(x)$
- Convolution with p.d.f.: $g * f(t) = \int_{-\infty}^{+\infty} g(t-x) f(x)\, dx$

Def. Function h is called locally bounded if

Def. Function h is absolutely integrable if

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies , then H is locally bounded

and

(b) Conversely, if H is a locally bounded solution to (*),

then

[convolution in the Riemann-Stieltjes sense]

(c) If h is absolutely integrable , then

No proof.

Remark. Key renewal theorem says that if h is locally bounded, then there exists a unique locally bounded solution to (*) given by (**)

Examples

- Renewal function: $M(t)$ satisfies

  and

  $F(t)$ is nondecreasing, so (c) does not apply to the renewal equation for $M(t)$

- Renewal density: $m(t)$ satisfies

  and

  (in the Riemann-Stieltjes sense)

  f is absolutely integrable, , so

Important remark

Let $W = (W_1, W_2, \dots)$ be arrival times of a renewal process, and denote $W' = (W_1', W_2', \dots)$ with

$$W_i' = W_{i+1} - W_1 = X_2 + X_3 + \dots + X_{i+1},$$

shifted arrival times.

Then:

- W'
- W'

Example

Example. Compute $\lim_{t\to\infty} E(\gamma_t)$. Take $H(t) = E(\gamma_t)$

If $X_1 > t$, then ; if $X_1 < t$ condition on $X_1 = s$

$E(\gamma_t) =$

$E(\gamma_t \mathbb{1}_{X_1 \le t}) =$

$=$

$=$

$=$

$=$

Example (cont)

Assume that $E(X_1)=\mu$, $Var(X_1)=\sigma^2$

$$E((X_1-t)\mathbb{1}_{X_1>t}) =$$

$$=$$

Since we assume that $E(X_1)=\sigma^2$,

and

Finally, we have that

$$H(t) =$$

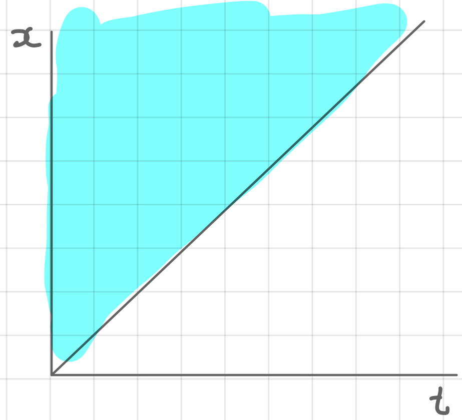
therefore $H(t) = h(t) + h * M(t)$

with $h(t) =$

Example (cont)

In particular,

$$\int_0^\infty \int_t^\infty (1 - F(x))\,dx\,dt =$$



$\Rightarrow$ by part (c) of the key renewal theorem

$$\lim_{t\to\infty} E(\gamma_t) =$$

Similarly $\lim_{t\to\infty} E(\delta_t) =$, $\lim_{t\to\infty} E(\beta_t) =$

Example

What is the expected time to the next earthquake in the long run?

For $X_1 \sim \text{Unif}[0,1]$

therefore, $\lim_{t \to \infty} E(\gamma_t) =$

And the long run expected time between two consecutive earthquakes is