MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Asymptotic behavior of renewal processes
Next: PK 7.5, Durrett 3.1, 3.3

Week 6:

- HW5 due Friday, May 12 on Gradescope

Limiting distribution of age and excess life

Assume that X_i are continuous. Then

$$P(\delta_t \geq x, \gamma_t > y) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u)) \, dF^{*k}(u)$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u)) \, d \overbrace{\sum_{k=1}^{\infty} F^{*k}(u)}^{M(t)}$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u)) \, m(u) \, du$$

$$=$$

Recall that $\varepsilon(s) := m(s) - \frac{1}{\mu} \to 0$ as $s \to \infty$ ($\mu = E(X_1)$). Then

$$\lim_{t \to \infty} P(\delta_t \geq x, \gamma_t > y) =$$

$$=$$

Joint/limiting distribution of (γ_t, δ_t)

Thm. Let $F(t)$ be the c.d.f. of the interrenewal times. Then

(a) $$P(\gamma_t > y, \delta_t \geq x) = 1 - F(t+y) + \sum_{k=1}^{\infty} \int_0^{t-x} (1 - F(t+y-u)) \, dF^{*k}(u)$$

$$= 1 - F(t+y) + \int_0^{t-x} (1 - F(t+y-u)) \, dM(u)$$

(b) if additionally the interrenewal times are continuous,

$$\lim_{t \to \infty} P(\gamma_t > y, \delta_t \geq x) = \frac{1}{\mu} \int_{x+y}^{\infty} (1 - F(w)) \, dw \quad (*)$$

If we denote by $(\gamma_\infty, \delta_\infty)$ a pair of r.v.s with distribution $(*)$ then γ_∞ and δ_∞ are continuous r.v.s with densities

$$f_{\gamma_\infty}(x) = f_{\delta_\infty}(x) =$$

Example

Renewal process (counting earthquakes in California) has interrenewal times uniformly distributed on $[0,1]$ (years).

(a) What is the long-run probability that an earthquake will hit California within 6 months?

(b) What is the long-run probability that it has been at most 6 months since the last earthquake?

Key renewal theorem

Suppose $H(t)$ is an unknown function that satisfies

$$H(t) = h(t) + H * F(t) \quad (*)$$

↑ renewal equation

E.g.: $M(t) = F(t) + M * F(t)$,

$$m(t) = f(t) + m * F(t) = f(t) + m * f(t)$$

Remark about notation

- Convolution with c.d.f.: $g * F(t) = \int_{-\infty}^{+\infty} g(t-x)\, dF(x)$
- Convolution with p.d.f.: $g * f(t) = \int_{-\infty}^{+\infty} g(t-x) f(x)\, dx$

Def. Function h is called locally bounded if

Def. Function h is absolutely integrable if

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies , then H is locally bounded

and

(b) Conversely, if H is a locally bounded solution to (*),

then [convolution in the Riemann-Stieltjes sense]

(c) If h is absolutely integrable, then

No proof.

Remark. Key renewal theorem says that if h is locally bounded, then there exists a unique locally bounded solution to (*) given by (**)

Examples

- Renewal function: $M(t)$ satisfies

  and

  $F(t)$ is nondecreasing, so (c) does not apply to the renewal equation for $M(t)$

- Renewal density: $m(t)$ satisfies

  and

  (in the Riemann-Stieltjes sense)

  f is absolutely integrable, , so

Important remark

Let $W=(W_1, W_2, \dots)$ be arrival times of a renewal process, and denote $W'=(W_1', W_i', \dots)$ with

$$W_i' = W_{i+1} - W_1 = X_2 + X_3 + \dots + X_{i+1},$$

shifted arrival times.

Then:

- W'
- W'

Example

Example. Compute $\lim_{t\to\infty} E(\gamma_t)$. Take $H(t) = E(\gamma_t)$; if $X_1 < t$ condition on $X_1 = s$

If $X_1 > t$, then

$E(\gamma_t) =$

$E(\gamma_t \mathbb{1}_{X_1 \le t}) =$

$=$

$=$

$=$

$=$

Example (cont)

Assume that $E(X_1)=\mu$, $Var(X_1)=\sigma^2$

$$E\left((X_1-t)\mathbb{1}_{X_1>t}\right) =$$

$$=$$

Since we assume that $E(X_1)=\sigma^2$,

and

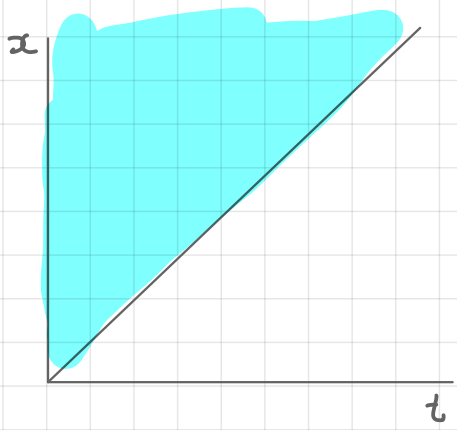
Finally, we have that

$$H(t) =$$

therefore $H(t) = h(t) + h * M(t)$

with $h(t) =$

Example (cont)



In particular,

$$\int_0^\infty \int_t^\infty (1 - F(x))\, dx\, dt =$$

$\Rightarrow$ by part (c) of the Key renewal theorem

$$\lim_{t \to \infty} E(\gamma_t) =$$

Similarly $\lim_{t \to \infty} E(\delta_t) =$, $\lim_{t \to \infty} E(\beta_t) =$

Example

What is the expected time to the next earthquake in the long run?

For $X_1 \sim \text{Unif}[0,1]$

therefore, $\lim_{t \to \infty} E(Y_t) =$

And the long run expected time between two consecutive earthquakes is