MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 7:

- HW6 due Friday, May 19 on Gradescope

Key renewal theorem

Thm (Key renewal theorem) Let h be locally bounded.

(a) If H satisfies $H = h + h * M$, then H is locally bounded and

$$H = h + H * F \qquad (*)$$

(b) Conversely, if H is a locally bounded solution to $(*)$, then

$$H = h + h * M \qquad (**)$$

[convolution in the Riemann-Stieltjes sense]

(c) If h is absolutely integrable, then

$$\lim_{t \to \infty} H(t) = \frac{\int_0^\infty h(x)\,dx}{\mu}$$

No proof.

Remark. Key renewal theorem says that if h is locally bounded, then there exists a unique locally bounded solution to $(*)$ given by $(**)$

Important remark

Let $W = (W_1, W_2, \ldots)$ be renewal times of a renewal process, and denote $W' = (W_1', W_2', \ldots)$ with

$$W_i' = W_{i+1} - W_1 = X_2 + X_3 + \cdots + X_{i+1},$$

shifted arrival times.

Then:

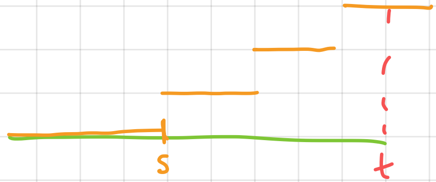
- W' is independent of $W_1 = X_1$
- W' has the same distribution as W

Example

Example. Compute $\lim_{t\to\infty} E(\gamma_t)$. Take $H(t) = E(\gamma_t)$

If $X_1 > t$, then $\gamma_t = X_1 - t$; if $X_1 < t$ condition on $X_1 = s$

$$E(\gamma_t) = E(\gamma_t \mathbb{1}_{X_1 > t}) + E(\gamma_t \mathbb{1}_{X_1 \le t})$$



$$E(\gamma_t \mathbb{1}_{X_1 \le t}) = \int^\infty P\left(\left(W_{N(t)+1} - t\right)\mathbb{1}_{X_1 \le t} > w\right) dw$$

$$= \int_0^\infty \sum_{k=1}^\infty P\left((W_k - t)\mathbb{1}_{X_1 \le t} > w \,,\ N(t) = k-1\right) dw$$

$$= \int_0^\infty \sum_{k=2}^\infty P\left(\left(X_1 + \sum_{j=2}^k X_j - t\right)\mathbb{1}_{X \le t} > w \,,\ N(t) = k-1\right) dw$$

$$= \int_0^\infty \sum_{k=2}^\infty \int_0^t P\left(\overbrace{\sum_{j=2}^k X_j}^{W'_{k-1}} - (t-s) > w \,,\ \overbrace{N(t) = \underbrace{k-1}_{= \ell}}^{N'(t-s) = k-2} \,\Big|\, X_1 = s\right) dF(s)\, dw$$

$$= \int_0^t \int_0^\infty \underbrace{\sum_{\ell=1}^\infty P\left(W'_\ell - (t-s) > w \,,\ N'(t-s) = \ell - 1\right)}_{= P(\gamma'_{t-s} > w)} dw\, dF(s) = \int_0^t \overbrace{E(\gamma'_{t-s})}^{H(t-s)} dF(s)$$

$$= H * F$$

Example (cont)

Assume that $E(X_1) = \mu$, $Var(X_1) = \sigma^2$

$$E\left((X_1 - t)\mathbb{1}_{X_1 > t}\right) = \int_t^\infty (x-t)\,dF(x) = \int_t^\infty (t-x)\,d(1-F(x))$$

$$= (t-x)(\underbrace{1-F(x)}_{0})\Big|_t^\infty + \int_t^\infty (1-F(x))\,dx$$

Since we assume that $Var(X_1) = \sigma^2$,

and Ex.: $x(1-F(x)) \to 0$ as $x \to \infty$

Finally, we have that

$$H(t) = \int_t^\infty (1-F(x))\,dx + H * F(t)$$

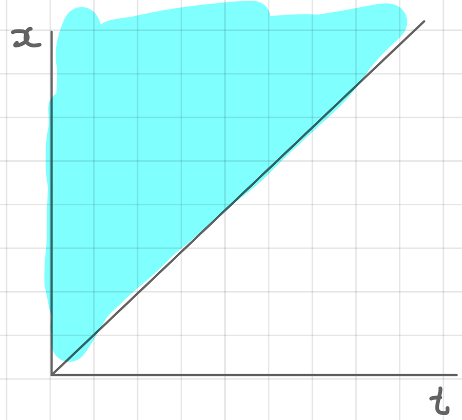
therefore $H(t) = h(t) + h * M(t)$

with $h(t) = \int_t^\infty (1-F(x))\,dx$

Example (cont)

In particular,

$$\int_0^\infty \int_t^\infty (1-F(x))\,dx\,dt = \int_0^\infty \int_0^x (1-F(x))\,dt\,dx$$



$$= \int_0^\infty x(1-F(x))\,dx = \frac{1}{2} E(X_i^2)$$

$$= \frac{1}{2}(\sigma^2 + \mu^2) < \infty \Rightarrow h(t) \text{ is absolutely integrable}$$

$\Rightarrow$ by part (c) of the key renewal theorem

$$\lim_{t\to\infty} E(\gamma_t) = \frac{1}{\mu} \cdot \frac{1}{2}(\sigma^2 + \mu^2)$$

Similarly $\displaystyle\lim_{t\to\infty} E(\delta_t) = \frac{\sigma^2+\mu^2}{2\mu}$, $\displaystyle\lim_{t\to\infty} E(\beta_t) = \frac{\sigma^2+\mu^2}{\mu} > \mu$

Example

What is the expected time to the next earthquake in the long run?

For $X_1 \sim \text{Unif}[0,1]$

$$E(X_1^2) = \int_0^1 x^2\,dx = \frac{1}{3} = \sigma^2 + \mu^2$$

therefore, $\lim_{t\to\infty} E(\gamma_t) = \dfrac{\sigma^2+\mu^2}{2\mu} = \dfrac{\frac{1}{3}}{2\cdot\frac{1}{2}} = \dfrac{1}{3}$

And the long run expected time between two consecutive earthquakes is $\frac{2}{3} > \frac{1}{2} = E(X_1)$

Remark: moments of nonnegative r.v.s

Proposition. Let X be a nonnegative random variable. Then

$$E(X^n) = n\int_0^\infty x^{n-1} P(X > x)\,dx$$

$$= n\int_0^\infty x^{n-1}(1 - F(x))\,dx$$

$n=2$:

$$E(X^2) = 2\int_0^\infty x(1-F(x))dx$$

Proof. $X \geq 0 \Rightarrow X^n \geq 0$. Using the "tail" formula for the expectation of nonnegative random variables

$$E(X^n) = \int_0^\infty P(X^n > t)\,dt = \int_0^\infty P(X > t^{1/n})\,dt$$

After the change of variable $x = t^{1/n}$ we get

$$E(X^n) = \int_0^\infty n x^{n-1} P(X > x)\,dx$$