MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Asymptotic behavior of renewal processes

Next: PK 7.5, Durrett 3.1, 3.3

Week 7:

- HW6 due Friday, May 19 on Gradescope

Remark. $M(t)$ is finite for all t

Proposition. Let $N(t)$ be a renewal process with interrenewal times X_i having distribution F. If there exist $c>0$ and $\alpha\in(0,1)$ such that $P(X_1>c)>\alpha$, then

Proof: Recall that $M(t)=\sum_{k=1}^{\infty}P(W_k\le t)=\sum_{k=1}^{\infty}P\left(\sum_{j=1}^{k}X_j\le t\right)$ $(*)$

Example: Age replacement policies (PK, p. 363)

Setting: - component's lifetime has distribution function F

- component is replaced

  (A) either when it fails,

  (B) or after reaching age T (fixed)

  whichever occurs first

- replacements (A) and (B) have different costs:

  replacement of a failed component (A) is more expensive than the planned replacement (B)

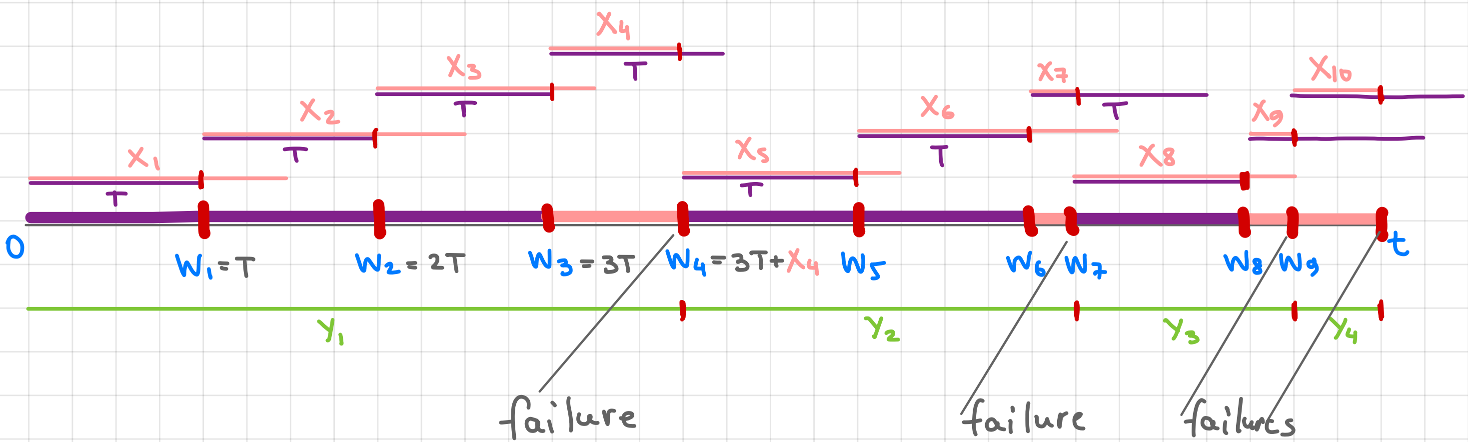
Question: How does the long-run cost of replacement depend on the cost of (A), (B) and age T?

What is the optimal T that minimizes the long-run cost of replacement?

Example: Age replacement policies (PK, p. 363)

Notation: X_i - lifetime of i-th component, $F_{X_i}(t) = F(t)$

Y_i - times between failures



Here we have two renewal processes

(1) renewal process $N(t)$ generated by renewal times $(W_i)_{i=1}^{\infty}$

(2) renewal process $Q(t)$ generated by interrenewal times $(Y_i)_{i=1}^{\infty}$

$N(t) =$, $Q(t) =$

Example: Age replacement policies (PK, p. 363)

Compute the distribution of the interrenewal times for $N(t)$

$$W_i - W_{i-1} = \begin{cases} \\ \\ \end{cases}, \text{ so}$$

$$F_T(x) := P(W_i - W_{i-1} \le x) = \begin{cases} \\ \\ \end{cases}$$

In particular,

$$E(W_i - W_{i-1}) =$$

Using the elementary renewal theorem for $N(t)$, the total number of replacements has a long-run rate

Example: Age replacement policies (PK, p. 363)

Compute the distribution of the interrenewal times for $Q(t)$.

$$Y_1 = \begin{cases} & \text{if } X_1 \le T \\ & \text{if } X_1 > T,\ X_2 \le T \\ \vdots & \\ & ,\ \text{if } X_1 > T, \ldots, X_n > T,\ X_{n+1} \le T \\ \vdots & \end{cases}$$

so

and for $z \in (0, T)$

$P(Z \le z) =$

$=$

$=$

Example: Age replacement policies (PK, p. 363)

Now we can compute the long-run rate of the replacements due to failures

$E(Y_1) =$

$E(L) =$

$E(Z) =$, so

$E(Y_1) =$

Applying the elementary renewal theorem to $Q(t)$

Example: Age replacement policies (PK, p. 363)

Suppose that the cost of one replacement is K, and each replacement due to a failure costs additional c. Then, in the long run the total amount spent on the replacements of the component per unit of time is given by

$$C(T) \approx$$

If we are given c, K and the distribution of the component's lifetime F, we can try to minimize the overall costs by choosing the optimal value of T.

Example: Age replacement policies (PK, p. 363)

For example, if $K=1$, $C=4$ and $X_1 \sim \text{Unif}[0,1]$ ($F(x) = x \mathbb{1}_{[0,1]}$)

For $T \in [0,1]$, $\mu_T =$ and

the average (per unit of time) long-run costs are

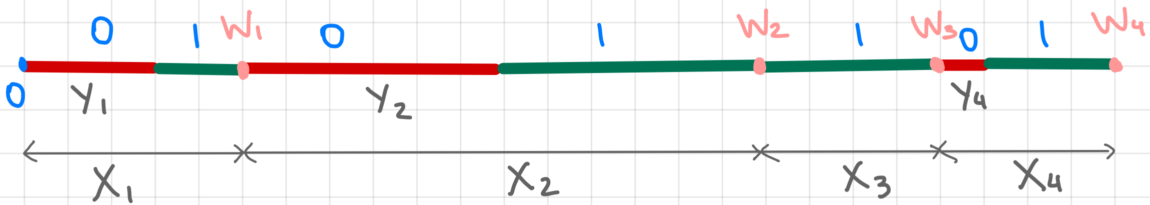
$$C(T) =$$

$$\frac{d}{dT} C(T) =$$

Two component renewals

Consider the following model:

- $(X_i)_{i=1}^{\infty}$ are interrenewal times
- at each moment of time the system $S(t)$ can be in one of two states: $S(t)=0$ or $S(t)=1$
- random variables Y_i denote the part of X_i during which the system is in state 0, $0 \le Y_i \le X_i$
- collection $((X_i, Y_i))_{i=1}^{\infty}$ is i.i.d.



Q: In the long run (for large t), what is the probability that the system is in state 1 at time t?

Two component renewals

<u>Thm</u>.

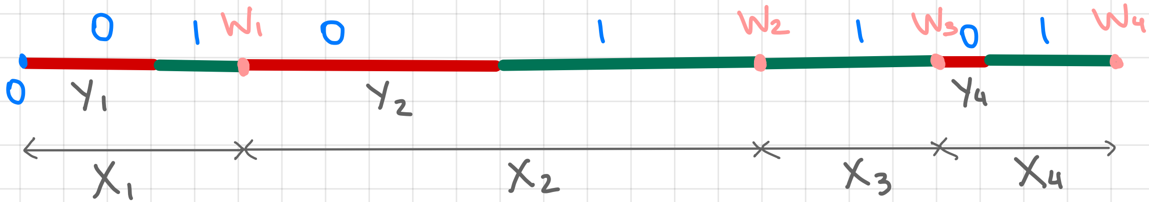
$$\lim_{t\to\infty} P(S(t)=0) =$$

<u>Proof</u>. Denote $g(t) =$. Then

$$g(t) =$$

If $t < x$, then $P(S(t)=0 \mid X_1 = x) =$

If $t \geq x$, then $P(S(t)=0 \mid X_1 = x) =$



Two component renewals

$$g(t) = \underbrace{\int_t^\infty \qquad\qquad\qquad\qquad}_{} + \underbrace{\int_0^t \qquad\qquad\qquad\qquad}_{}$$

Function g satisfies the renewal equation

$$g(t) =$$

Note that $Y_1 \le X_1$, therefore $P(Y_1 > t \mid X_1 = x) =$ for $x < t$,

$$h(t) =$$

$$\int_0^\infty h(t)\,dt =$$

From the Key renewal theorem $\lim_{t\to\infty} g(t) =$

Example: the Peter principle

Setting:
- infinite population of candidates for certain position
- fraction p of the candidates are competent, $q = 1-p$ are incompetent
- if a competent person is chosen, after time C_i he/she gets promoted
- if an incompetent person is chosen, he/she remains in the job until retirement (r.v. I_j)
- once the position is open again, the process repeats

Question: What fraction of time, denoted f, is the position held by an incompetent person on average in the long run?

Example: the Peter principle

Denote $X_i = \begin{cases} \quad , & \text{if occupied by a competent person} \\ \quad , & \text{if occupied by an incompetent person} \end{cases}$

$Y_i = \begin{cases} \quad , & \text{if occupied by a competent person} \\ \quad , & \text{if occupied by an incompetent person} \end{cases}$

KRT for two component renewals can be applied to $((X_i, Y_i))_{i=1}^{\infty}$

If $S(t) = 0$ if the person is incompetent, then

$$\lim_{t \to \infty} P(S(t) = 0) = \frac{E(Y_1)}{E(X_1)} \qquad \text{and}$$

$$f := \lim_{t \to \infty} \left(\qquad \qquad \right) = \lim_{t \to 0} \frac{1}{t} \int_0^t \qquad du =$$

Finally, if
- $E(C_i) = \mu$
- $E(I_i) = \nu$

then $f = \dfrac{E(Y_1)}{E(X_1)} = \dfrac{(1-p)\nu}{p\mu + (1-p)\nu}$

Example: the Peter principle (alternative)

Let $X_i = \begin{cases} C_i, & \text{if the i-th person is competent} \\ I_i, & \text{if the i-th person is incompetent} \end{cases}$

$Y_i = \begin{cases} 0, & \text{time occupied by a competent person} \\ I_i, & \text{time occupied by an incompetent person} \end{cases}$

and assume that $|X_i| < K$. Then using

$$\leq E\left(\frac{1}{t}\int_0^t \mathbb{1}_{\{S(u)=0\}}\,du\right) \leq$$

Again, if
- $E(C_i) = \mu$
- $E(I_i) = \nu$

, then $f = \dfrac{E(Y_1)}{E(X_1)} = \dfrac{(1-p)\nu}{p\mu + (1-p)\nu}$

Example: the Peter principle

If we take , then

$$f =$$