MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Introduction. Birth processes

Next: PK 6.2-6.3

Week 1:

- visit course web site
- join Piazza

Stochastic (random) processes

Def. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Stochastic process is a collection $(X_t : t \in T)$ of random variables (all defined on the same probability space)

- often t represents time, but can be 1-D space
- T is called the index set, S is called the state space
- $X : \Omega \times T \to S$ $(X_t(\omega) \in S)$
- for any fixed ω, we get a realization of all random variables $(X_t(\omega) : t \in T)$ ← sample path trajectory

- stochastic process

Stochastic processes. Classification

Questions:

- What is T
- What is S
- Relations between X_{t_1} and X_{t_2} for $t_1 \neq t_2$?
- Properties of the trajectory

Discrete time

$T = \mathbb{N}, \mathbb{Z},$ finite set

↳ random vector

Continuous time

$T = \mathbb{R}, [0, +\infty), [0, 1]$

Real-valued

$S = \mathbb{R}$

Integer-valued

$S = \mathbb{Z}$

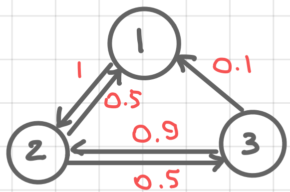
Nonnegative ...

$S \subset [0, +\infty)$

Continuous, right-continuous (cadlag) sample path

Examples of stochastic processes

- Gaussian processes: for any $t \in T$, X_t has normal distrib.
- Stationary processes: distribution doesn't change in time
- Processes with stationary and independent increments (Levy
- Poisson process: increments are independent and Poisson$(\cdot)$
- Markov processes: "distribution in the future depends only on the current state, but does not depend on the past"

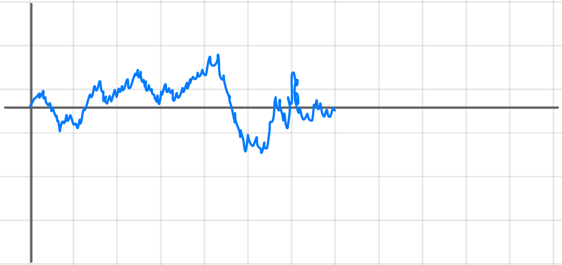


Examples of stochastic processes

- Martingales : $\mathbb{E}[X_{n+1} \mid X_n, \dots, X_1, X_0] = X_n$ ("fair game")

- Brownian motion (Wiener process) is a continuous-time st. proc.
  Gaussian, martingale, has stationary and independent increments, Markov, $\mathrm{Var}[W_t] = t$
  $\mathrm{Cov}[W_t, W_s] = \min\{s,t\}$, its sample path is everywhere continuous and nowhere differentiable

- diffusion processes (stochastic differential equations)



- ....

Continuous time MC

Continuous Time Markov Chains

<u>Def</u> (Discrete-time Markov chain)

Let $(X_n)_{n\geq 0}$ be a discrete time stochastic process taking values in $\mathbb{Z}_+ = \{0,1,2,\dots\}$ (for convenience). $(X_n)_{n\geq 0}$ is called Markov chain if for any $n \in \mathbb{N}$ and $i_0, i_1, \dots, i_{n-1}, i, j \in \mathbb{Z}_+$

$$P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i) = P(X_{n+1} = j \mid X_n = i)$$

<u>Def</u> (Continuous-time Markov chain)

Let $(X_t)_{t\geq 0} = (X_t : 0 \leq t < \infty)$ be a continuous time process taking values in $\mathbb{Z}_+$. $(X_t)_{t\geq 0}$ is called Markov chain if for any $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \dots < t_{n-1} < s$, $t > 0$, $i_0, i_1, \dots, i_{n-1}, i, j \in \mathbb{Z}_+$

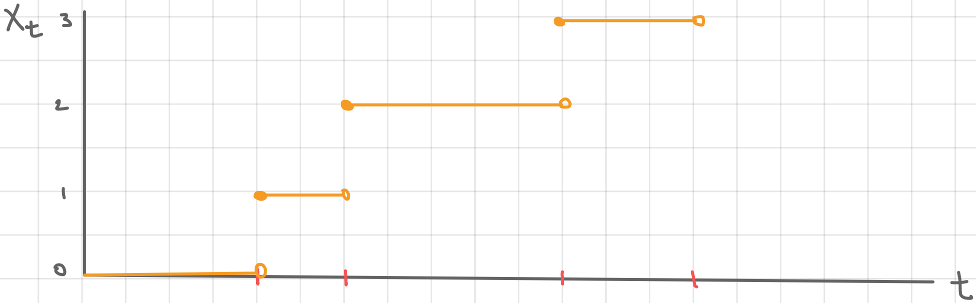
Poisson process

<u>Def</u>. A Poisson process of intensity (rate) $\lambda > 0$ is an integer-valued stochastic process $(X_t)_{t \geq 0}$ for which

1) for each time points $t_0 = 0 < t_1 < \cdots < t_n$, the process increments $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent random variables

2) for $s \geq 0$ and $t > 0$, the random variable $X_{s+t} - X_s$ has the Poisson distribution $\mathbb{P}[X_{t+s} - X_s = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, $k = 0, 1, \dots$

3) $X_0 = 0$



Example: Poisson process as MC

Is Poisson process a continuous time MC?

Poisson process:

continuous time

discrete state

(*)

Let $(X_t)_{t\geq 0}$ be a Poisson process, let $i_0 \leq i_1 \leq \cdots \leq i_{n-1} \leq i \leq j$

$$P(X_{s+t}=j \mid X_{t_0}=i_0, X_{t_1}=i_1, \ldots, X_{t_{n-1}}=i_{n-1}, X_s=i)$$

$=$

$=$

$=$

Transition probability function

One way of describing a continuous time MC is by using the transition probability functions.

Def. Let $(X_t)_{t\geq 0}$ be a MC. We call

$$i,j \in \{0,1,\dots\}, \quad s \geq 0, \quad t > 0$$

the transition probability function for $(X_t)_{t\geq 0}$.

If $P(X_{s+t} = j \mid X_s = i)$ does not depend on s, we say that $(X_t)_{t\geq 0}$ has stationary transition probabilities and we define

[compare with n-step transition probabilities]

Characterization of the Poisson process

Experiment: count events occurring along $[0,+\infty)$ $\left\{\begin{array}{l}\text{time} \\ \text{or 1-D space}\end{array}\right.$

Denote by $N((a,b])$ the number of events that occur on $(a,b]$.

Assumptions:

1. Number of events happening in disjoint intervals are independent.
2. For any $t \geq 0$ and $h > 0$, the distribution of $N((t,t+h])$ does not depend on t (only on h, the length of the interval)
3. There exists $\lambda > 0$ s.t. $P(N((t,t+h]) \geq 1) = \lambda h + o(h)$ as $h \to 0$
   (rare events)
4. Simultaneous events are not possible: $P(N((t,t+h]) \geq 2) = o(h)$, $h \to 0$

Transition probabilities of the Poisson process

Let $(X_t)_{t\geq 0}$ be the Poisson process.

Define the transition probability functions

$$P_{ij}(h) := P(X_{t+h} = j \mid X_t = i), \quad i,j \in \{0,1,2,\dots\}, \quad t \geq 0, \ h > 0$$

What are the infinitesimal (small h) transition probability functions for $(X_t)_{t\geq 0}$? As $h \to 0$,

$$P_{ii}(h) = P(X_{t+h} = i \mid X_t = i)$$

$$=$$

$$P_{i,i+1}(h) = P(X_{t+h} = i+1 \mid X_t = i) =$$

$$\sum_{j \notin \{i,i+1\}} P_{ij}(h) =$$

Poisson process and transition probabilities

To sum up: $(X_t)_{t\geq 0}$ is a MC with (infinitesimal) transition probabilities satisfying

$$P_{ii}(h) =$$

$$P_{i,i+1}(h) =$$

$$\sum_{j\notin\{i,i+1\}} P_{i,j}(h) =$$

What if we allow $P_{ij}(h)$ depend on i?

↳ birth and death processes

Pure birth processes

Def Let $(\lambda_k)_{k\geq 0}$ be a sequence of positive numbers.
We define a pure birth process as a Markov process $(X_t)_{t\geq 0}$ whose stationary transition probabilities satisfy

1. $P_{k,k+1}(h) =$
2. $P_{k,k}(h) =$
3. $P_{k,j}(h) =$
4. $X_0 = 0$

Related model. Yule process : $\lambda_k = \beta k$ for some $\beta > 0$.
Describes the growth of a population
- birth rate is proportional to the size of the population

Birth processes and related differential equations

Now define $P_n(t) = P(X_t = n)$. For small $h > 0$

$P_n(t+h) = P(X_{t+h} = n) =$

$=$

$=$

$=$

$=$

$$P_n(t+h) - P_n(t) = -\lambda_n h P_n(t) + \lambda_{n-1} h P_{n-1}(t) + o(h)$$

$P_n'(t) =$

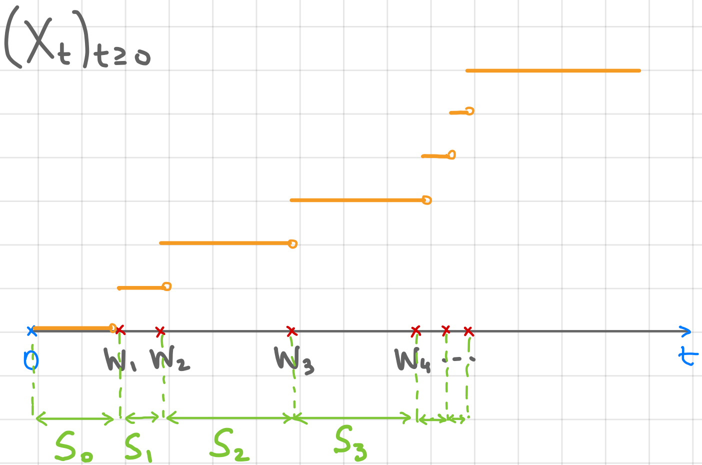
Birth processes and related differential equations

$P_n(t)$ satisfies the following system of differentian eqs. with initial conditions

$$
(*) \left\{ \begin{array}{l} P_0'(t) = \\ P_1'(t) = \\ P_2'(t) = \\ \vdots \\ P_n'(t) = \\ \vdots \end{array} \right.
\qquad
\begin{array}{l} P_0(0) = \\ P_1(0) = \\ P_2(0) = \\ \vdots \\ P_n(0) = \\ \vdots \end{array}
$$

Solving this system gives the p.m.f. of X_t for any t

Description of the birth processes via sojourn times



W_i – i-th "birth time" $\quad$ S_i – "time between (i-1)-th birth and i-th birth"

↳ sojourn times

$$W_i = \sum_{l=0}^{i-1} S_l$$

Alternative way of characterizing $(X_t)_{t\geq 0}$:

–

–

Description of the birth processes via sojourn times

Theorem

Let $(\lambda_k)_{k \geq 0}$ be a sequence of positive numbers. Let $(X_t)_{t \geq 0}$ be a non-decreasing right-continuous process, $X_0 = 0$, taking values in $\{0, 1, 2 \dots\}$. Let $(S_i)_{i \geq 0}$ be the sojourn times associated with $(X_t)_{t \geq 0}$, and define $W_\ell = \sum_{i=0}^{\ell-1} S_i$.

Then conditions

(a)

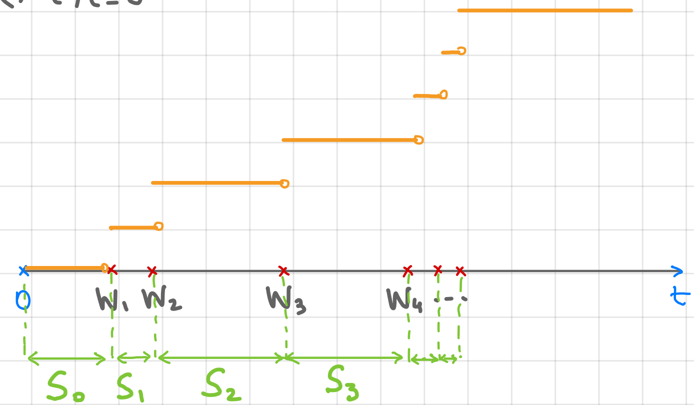
(b)

are equivalent to

(c)

Explosion

$(X_t)_{t\geq 0}$



population becomes infinite in finite time

Thm. Let $(X_t)_{t\geq 0}$ be a pure birth process of rates $(\lambda_k)_{k\geq 0}$.

Then

Solving the system of differential equations (*)

$$(*) \begin{cases} P_0'(t) = -\lambda_0 P_0(t), & P_0(0) = 1 \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), & P_n(0) = 0 \quad \text{for } n \geq 1 \end{cases}$$

$P_0(t)$:

$$P_0'(t) =$$

$$\frac{P_0'(t)}{P_0(t)} =$$

$$g'(t) =$$

$$g(t) =$$

Solving the system of differential equations (*)

$P_n(t)$, $n \geq 1$

Consider the function $Q_n(t) =$

$(Q_n(t))' =$

$Q_n(t) =$

$\hookrightarrow P_n(t) =$ $\leftarrow$ apply recursively

$$P_1(t) = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{\lambda_1 s} e^{-\lambda_0 s} ds = e^{-\lambda_1 t} \int_0^t \lambda_0 e^{(\lambda_1 - \lambda_0)s} ds \quad (\text{if } \lambda_1 \neq \lambda_0)$$

$$= e^{-\lambda_1 t} \frac{\lambda_0}{\lambda_1 - \lambda_0} \left(e^{(\lambda_1 - \lambda_0)t} - 1 \right) = \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 t} - \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_1 t}$$