MATH180C: Introduction to Stochastic Processes II https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Martingales

Next: PK 2.5, Durrett 5.1-5.2

Week 8:

HW6 due Monday, May 22 on Gradescope

Midterm 2 on Wednesday, May 24

Martingales

Definition. A stochastic process (Xn, n 20) is a

martingale if for n=0.1,...

- (a) $E(|X_n|) < \infty$
- (b) $E(X_{n+1} | X_0, X_1, ..., X_n) = X_n$

After taking the expectation of both sides of (b)

we get that $E(X_{n+1}) = E(X_n)$

(Xn)nzo is a martingale => E(Xn) = E(Xo) Yn

- submartingale : E(Xn+1 | Xo,..., Xn) 2 Xn (increases)
- · supermartingale: E(Xn+1/Xo,..., Xn) = Xn (decreases)

Examples of martingales

(i) Let X1, X2, ... be independent RV's with E(IXKI) <~ and $E(X_k) = 0$. Define $S_n = X_1 + \dots + X_n$, $S_n = 0$. Then $E(S_{n+1}|S_{0},...,S_{n}) = E(S_{n} + X_{n+1}|S_{0},...,S_{n})$ = E (Sn | So, ..., Sn) + E (Xn+1 | So, ..., Sn) $= S_n + E(X_{n+1}) = S_n$ => $(5n)_{n\geq 0}$ is a martingale with E(5n)=0(ii) Let X1, X2,... be independent RV with XKZO, E (IXXI) <∞ and $E(X_k)=1$. Define $M_n = X_1 X_2 \cdots X_n$, $M_o=1$. Then $E(M_{n+1} | M_{01}, M_n) = E(M_n \cdot X_{n+1} | M_{01}, M_n)$ $= M_n E(X_{n+1} | M_{0,--}, M_n) = M_n \cdot E(X_{n+1}) = M_n$ => (Mn) nzo is a martingale with E (Mn) = E(Mo)=1

Example

Stock prices in a perfect market

Let Xn be the closing price at the end of day n

of a certain publicly traded security such as a share or

stock. Many scholars believe that in a perfect

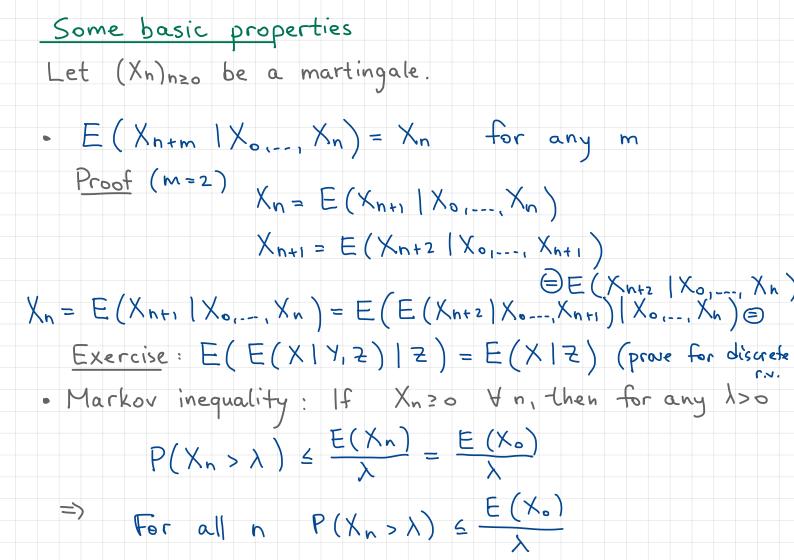
market these price sequences should be martingales.

(see PK page 73 for more details).

History and gambling

Let (Xn)nzo be a stochastic process describing your total winnings in n games with unit stake. Think of Xn-Xn-1 as your net winnings per unit stake in game n, n≥1, in a series of games, played at times n=1,2,.... In the martingale case E(Xn-Xn-, IXo, X1, ---, Xn-1) = E (Xn | Xo, ---, Xn-,) - E (Xn-, | Xo, --, Xn-,) $= X_{n-1} - X_{n-1} = O \quad (fair game)$

Some early works of martingales was motivated by gambling. Note that there exists a betting strategy called the "martingale system" < doubling bets after losses



Maximal inequality for nonegative martingales Thm. Let (Xn)n≥o be a martingale with nonnegative values. For any 2>0 and men $P\left(\begin{array}{c}\max X_n \geqslant \lambda\right) \leq \frac{E(X_o)}{\lambda}$ (1) and $P(\max_{n\geq 0} X_n \geq \lambda) \leq \frac{E(X_0)}{\lambda}$ (2) Proof. We prove (1), (2) follows by taking the limit m+. Take the vector (Xo, XI, --, Xm) and partition the sample space wrt the index of the first r.v. rising above) $1 = 1_{X_0 \ge \lambda} + 1_{X_0 < \lambda, X_1 \ge \lambda} + \cdots + 1_{X_0 < \lambda, \cdots, X_{m-1} < \lambda, X_m \ge \lambda} + 1_{X_0 < \lambda, \cdots, X_{m < \lambda}}$ Compute E(Xm) = E(Xm. 1) using the above partition

