MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Martingales

Next: PK 2.5, Durrett 5.1-5.2

Week 8:

- HW6 due Monday, May 22 on Gradescope
- Midterm 2 on Wednesday, May 24

Martingales

Definition. A stochastic process $(X_n, n \geq 0)$ is a martingale if for $n = 0, 1, \ldots$

(a) $E(|X_n|) < \infty$

(b) $E(X_{n+1} \mid X_0, X_1, \ldots, X_n) = X_n$

After taking the expectation of both sides of (b) we get that

$$E(X_{n+1}) = E(X_n)$$

$(X_n)_{n \geq 0}$ is a martingale $\Rightarrow E(X_n) = E(X_0) \ \forall n$

- submartingale: $E(X_{n+1} \mid X_0, \ldots, X_n) \geq X_n$ (increases)
- supermartingale: $E(X_{n+1} \mid X_0, \ldots, X_n) \leq X_n$ (decreases)

Examples of martingales

(i) Let $X_1, X_2, \dots$ be independent RV's with $E(|X_k|) < \infty$ and $E(X_k) = 0$. Define $S_n = X_1 + \dots + X_n$, $S_0 = 0$.

Then $E(S_{n+1} \mid S_0, \dots, S_n) = E(S_n + X_{n+1} \mid S_0, \dots, S_n)$

$$= E(S_n \mid S_0, \dots, S_n) + E(X_{n+1} \mid S_0, \dots, S_n)$$

$$= S_n + E(X_{n+1}) = S_n$$

$\Rightarrow$ $(S_n)_{n \geq 0}$ is a martingale with $E(S_n) = 0$

(ii) Let $X_1, X_2, \dots$ be independent RV with $X_k \geq 0$, $E(|X_k|) < \infty$ and $E(X_k) = 1$. Define $M_n = X_1 X_2 \cdots X_n$, $M_0 = 1$.

Then $E(M_{n+1} \mid M_0, \dots, M_n) = E(M_n \cdot X_{n+1} \mid M_0, \dots, M_n)$

$$= M_n E(X_{n+1} \mid M_0, \dots, M_n) = M_n \cdot E(X_{n+1}) = M_n$$

$\Rightarrow$ $(M_n)_{n \geq 0}$ is a martingale with $E(M_n) = E(M_0) = 1$

Example

Stock prices in a perfect market

Let X_n be the closing price at the end of day n of a certain publicly traded security such as a share or stock. Many scholars believe that in a perfect market these price sequences should be martingales. (see PK page 73 for more details).

History and gambling

Let $(X_n)_{n\geq 0}$ be a stochastic process describing your total winnings in n games with unit stake.

Think of $X_n - X_{n-1}$ as your net winnings per unit stake in game n, $n \geq 1$, in a series of games, played at times $n = 1, 2, \ldots$.

In the martingale case $E(X_n - X_{n-1} \mid X_0, X_1, \ldots, X_{n-1})$

$$= E(X_n \mid X_0, \ldots, X_{n-1}) - E(X_{n-1} \mid X_0, \ldots, X_{n-1})$$

$$= X_{n-1} - X_{n-1} = 0 \quad (\text{fair game})$$

Some early works of martingales was motivated by gambling. Note that there exists a betting strategy called the "martingale system" ← doubling bets after losses

Some basic properties

Let $(X_n)_{n\geq 0}$ be a martingale.

- $E(X_{n+m} \mid X_0, \dots, X_n) = X_n$ for any m

<u>Proof</u> $(m=2)$

$$X_n = E(X_{n+1} \mid X_0, \dots, X_n)$$

$$X_{n+1} = E(X_{n+2} \mid X_0, \dots, X_{n+1})$$

$$\stackrel{\circ}{=} E(X_{n+2} \mid X_0, \dots, X_n)$$

$$X_n = E(X_{n+1} \mid X_0, \dots, X_n) = E(E(X_{n+2} \mid X_0, \dots, X_{n+1}) \mid X_0, \dots, X_n) \stackrel{\circ}{=}$$

<u>Exercise</u>: $E(E(X \mid Y, Z) \mid Z) = E(X \mid Z)$ (prove for discrete r.v.

- Markov inequality: If $X_n \geq 0$ $\forall n$, then for any $\lambda > 0$

$$P(X_n > \lambda) \leq \frac{E(X_n)}{\lambda} = \frac{E(X_0)}{\lambda}$$

$\Rightarrow$ For all n $\quad P(X_n > \lambda) \leq \dfrac{E(X_0)}{\lambda}$

Maximal inequality for nonnegative martingales

Thm. Let $(X_n)_{n\geq 0}$ be a martingale with nonnegative values.
For any $\lambda > 0$ and $m \in \mathbb{N}$

$$P\left(\max_{0\leq n\leq m} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (1)$$

and

$$P\left(\max_{n\geq 0} X_n \geq \lambda\right) \leq \frac{E(X_0)}{\lambda} \qquad (2)$$

Proof. We prove (1), (2) follows by taking the limit $m\to\infty$.
Take the vector $(X_0, X_1, \dots, X_m)$ and partition the sample space wrt the index of the first r.v. rising above λ

$$1 = \mathbb{1}_{X_0\geq\lambda} + \mathbb{1}_{X_0<\lambda, X_1\geq\lambda} + \dots + \mathbb{1}_{X_0<\lambda,\dots,X_{m-1}<\lambda, X_m\geq\lambda} + \mathbb{1}_{X_0<\lambda,\dots,X_m<\lambda}$$

Compute $E(X_m) = E(X_m \cdot 1)$ using the above partition

Proof of the maximal inequality

$$E(X_m) = \sum_{n=0}^{m} E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda}) + E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_m<\lambda})$$

$$\geq \sum_{n=0}^{m} E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda})$$

Compute $E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda})$ by conditioning on $X_0, X_1, \dots, X_{n-1}, X_n$:

$$E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda})$$

$$= E(E(X_m \mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda} \mid X_0,\dots,X_n))$$

$$= E(\mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda} E(X_m \mid X_0,\dots,X_n))$$

$$= E(\mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda} X_n) \geq \lambda E(\mathbb{1}_{X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda})$$

Sum for all n

$$E(X_m) \geq \lambda \sum_{n=0}^{m} P(X_0<\lambda,\dots,X_{n-1}<\lambda,X_n\geq\lambda) = \lambda P(\max_{0\leq n\leq m} X_n \geq \lambda)$$