MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Brownian motion

Next: PK 8.1- 8.2

Week 9:

- homework 7 (due Friday, June 2)

Brownian motion. History

- Critical observation: Robert Brown (1827), botanist, movement of pollen grains in water
- First (?) mathematical analysis of Brownian motion: Louis Bachelier (1900), modeling stock market fluctuations
- Brownian motion in physics: Albert Einstein (1905) and Marian Smoluchowski (1906), explained the phenomenon observed by Brown
- First rigorous construction of mathematical Brownian motion: Norbert Wiener (1923)

Brownian motion $\underset{\text{in mathematics}}{=}$ Wiener process

Brownian motion. Motivation

- almost all interesting classes of stochastic processes contain Brownian motion: BM is a
  - martingale
  - Markov process
  - Gaussian process
  - Lévy process (independent stationary increments)
- BM allows explicit calculations, which are impossible for more general objects
- BM can be used as a building block for other processes
- BM has many beautiful mathematical properties

Brownian motion. Definition

Def. **Brownian motion** with diffusion coefficient σ^2 is a continuous time stochastic process $(B_t)_{t\geq 0}$ satisfying

(i) $B(0) = 0$, $B(t)$ is continuous as a function of t

(ii) For all $s<t$ $B(t)-B(s)$ is a Gaussian random variable with mean 0 and variance $\sigma^2(t-s)$

(iii) The increments of B are independent: if $0 \leq t_0 < t_1 < \dots < t_n$ then $\{B(t_i)-B(t_{i-1})\}_{i=1}^{n}$ are independent (Gaussian) rvs

$\sigma^2 = 1 \leftarrow$ standard BM

BM as a continuous time continuous space Markov process

Recall: continuous time discrete space MC $(X_t)_{t\geq 0}$ is characterized by the transition probability function

$$P_{ij}(t) = P(X_{t+s} = j \mid X_s = i)$$

($(X_t)_{t\geq 0}$ has stationary transition probability functions)

In particular, $P(X_{s+t} \in A \mid X_s = i) = \sum_{j\in A} P_{ij}(t)$

In the continuous state space case the transition probabilities are described by the transition density

(i) $p_t(x,y) \geq 0$, $\int_{-\infty}^{+\infty} p_t(x,y)\,dy = 1$ for all t, x

(ii) $P(X_{s+t} \in A \mid X_s = x) = \int_A p_t(x,y)\,dy$ for any $x \in \mathbb{R}$, $A \subset \mathbb{R}$

$\uparrow$ density of X_{s+t} given $X_s = x$

BM as a continuous time continuous space Markov process

<u>Propotition.</u> Let $(B_t)_{t\geq 0}$ be a standard BM. Then $(B_t)_{t\geq 0}$ is a Markov process with transition density

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-y)^2}$$

<u>Informal explanation</u>: Independent stationary increments imply that $(B_t)_{t\geq 0}$ is Markov with stationary transition density. Given $B_s = x$, $B_{t+s} = B_s + (B_{t+s} - B_s)$ information before time s is irrelevant.

$$P(B_{s+t} \leq u \mid B_s = x) = P(B_s + (B_{t+s} - B_s) \leq u \mid B_s = x)$$

$$= P(x + B_{t+s} - B_s \leq u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy$$

BM as a continuous time continuous space Markov process

Let $t_1 < t_2 < \cdots < t_n < \infty$, $(a_i, b_i) \subset \mathbb{R}$. Then

$$P\left(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2)\right) =$$

$$= \int_{-\infty}^{+\infty} P\left(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2) \mid B_{t_1} = x_1\right) p_{t_1}(0, x_1)\, dx_1$$

$$= \int_{a_1}^{b_1} P\left(B_{t_2} \in (a_2, b_2) \mid B_{t_1} = x_1\right) p_{t_1}(0, x_1)\, dx_1$$

$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} p_{t_2 - t_1}(x_1, x_2)\, dx_2\, p_{t_1}(0, x_1)\, dx_1$$

More generally,

$$P\left(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2), \ldots, B_{t_n} \in (a_n, b_n)\right)$$

$$= \int \cdots \int_{(a_1, b_1) \times \cdots \times (a_n, b_n)} p_{t_1}(0, x_1)\, p_{t_2 - t_1}(x_1, x_2) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n)\, dx_1 \cdots dx_n$$

Diffusion equation. Transition semigroup. Generator

Let $(X_t)_{t\geq 0}$ be a Markov process,

Suppose we want to know how the distribution of X_t evolves in time:

$$\lim_{t\downarrow 0} \text{"}\frac{P_t - \mathrm{Id}}{t}\text{"}$$

$$E\left(f\left(X_{t+s}\right) \mid X_s = x\right) = \int_{-\infty}^{+\infty} f(y)\, p_t^X(x,y)\, dy =: P_t f(x)$$

We call $(P_t)_{t\geq 0}$ the transition semigroup $\left[P_{s+t} f(x) \overset{CK}{=} P_s(P_t f(x))\right]$

Proposition Let $(P_t)_{t\geq 0}$ be the transition semigroup of BM.

Then (i) the "infinitesimal generator" of $P(t)$ is given by

$$Qf(x) = \frac{1}{2}\frac{d^2}{dx^2} f(x)$$

(ii) density p_t satisfies $\frac{\partial}{\partial t} p_t(x,y) = \frac{1}{2}\frac{\partial^2}{\partial x^2} p_t(x,y)$ [K backward]

(iii) density p_t satisfies $\frac{\partial}{\partial t} p_t(x,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2} p_t(x,y)$ [K forward]

diffusion equation, heat equation