MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Brownian motion

Next: PK 8.1- 8.2

Week 9:

- homework 6 (due Friday, June 2)

Brownian motion. History

- Critical observation: Robert Brown (1827), botanist, movement of pollen grains in water
- First (?) mathematical analysis of Brownian motion: Louis Bachelier (1900), modeling stock market fluctuations
- Brownian motion in physics: Albert Einstein (1905) and Marian Smoluchowski (1906), explained the phenomenon observed by Brown
- First rigorous construction of mathematical Brownian motion: Norbert Wiener (1923)

Brownian motion $\underset{\substack{\uparrow \\ \text{in mathematics}}}{=}$ Wiener process

Brownian motion. Motivation

- almost all interesting classes of stochastic processes contain Brownian motion: BM is a
  - martingale
  - Markov process
  - Gaussian process
  - Lévy process (independent stationary increments)
- BM allows explicit calculations, which are impossible for more general objects
- BM can be used as a building block for other processes
- BM has many beautiful mathematical properties

Brownian motion. Definition

Def. **Brownian motion** with diffusion coefficient σ^2 is a continuous time stochastic process $(B_t)_{t\geq 0}$ satisfying

(i)

(ii)

(iii)

$\sigma^2 = 1 \leftarrow$ standard BM

BM as a continuous time continuous space Markov process

<u>Recall:</u> continuous time discrete space MC $(X_t)_{t\geq 0}$ is characterized by the transition probability function

$$P_{ij}(t) =$$

($(X_t)_{t\geq 0}$ has stationary transition probability functions)

In particular, $P(X_{s+t} \in A \mid X_s = i) =$

In the continuous state space case the transition probabilities are described by the transition density

(i)

(ii) $P(X_{s+t} \in A \mid X_s = x) =$ for any $x \in \mathbb{R}$, $A \subset \mathbb{R}$

density of X_{s+t} given $X_s = x$

BM as a continuous time continuous space Markov process

<u>Propotition.</u> Let $(B_t)_{t\geq 0}$ be a standard BM.
Then $(B_t)_{t\geq 0}$ is a with transition density

<u>Informal explanation</u>: Independent stationary increments imply that $(B_t)_{t\geq 0}$ is Markov with stationary transition density. Given $B_s = x$,
information before time s is irrelevant.

$$P(B_{s+t} \leq u \mid B_s = x) =$$

BM as a continuous time continuous space Markov process

Let $t_1 < t_2 < \cdots < t_n < \infty$, $(a_i, b_i) \subset \mathbb{R}$. Then

$P(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2)) =$

$=$

$=$

$=$

More generally,

$$P(B_{t_1} \in (a_1, b_1), B_{t_2} \in (a_2, b_2), \dots, B_{t_n} \in (a_n, b_n))$$

$$= \int_{(a_1, b_1) \times \cdots \times (a_n, b_n)} \!\!\cdots \int p_{t_1}(0, x_1)\, p_{t_2 - t_1}(x_1, x_2) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n)\, dx_1 \cdots dx_n$$

Diffusion equation. Transition semigroup. Generator

Let $(X_t)_{t\geq 0}$ be a Markov process,

Suppose we want to know how the distribution of X_t evolves in time :

We call $(P_t)_{t\geq 0}$ the transition semigroup $\left[P_{s+t} f(x) \overset{\text{CK}}{=} P_s(P_t f(x)) \right]$

<u>Proposition</u> Let $(P_t)_{t\geq 0}$ be the transition semigroup of BM.

Then (i) the "infinitesimal generator" of $P(t)$ is given by

(ii) density p_t satisfies [K backward]

(iii) density p_t satisfies [K forward]

diffusion equation

BM as a Gaussian process

<u>Def</u>. Stochastic process $(X_t)_{t\geq 0}$ is called a Gaussian process if for any $0 \leq t_1 < t_2 < \dots < t_n$

$(X_{t_1}, \dots, X_{t_n})$ is a Gaussian vector, or equivalently for any $c_1, \dots, c_n \in \mathbb{R}$

is a Gaussian r.v.

Recall that the distribution of a Gaussian vector is uniquelly defined by its mean and covariance matrix.

Similarly, each Gaussian process is uniquely described by

$$\mu(t) = E(X_t) \quad \text{and} \quad \Gamma(s,t) = \mathrm{Cov}(X_s, X_t) \geq 0$$

↑ covariance function

BM as a Gaussian process

<u>Proposition</u> BM is a Gaussian process with

and

<u>Proof</u>. For any $0 \leq t_1 < t_2 < \cdots < t_n$, $B_{t_j} - B_{t_{j-1}}$ are indep. Gaussian, thus $\sum_{i=1}^{n} c_i B_{t_i} =$

is also Gaussian.

By definition . Let $s < t$.

Then $\Gamma(s,t) =$

$=$

$=$

$=$

Some properties of BM

Proposition. Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i) For any $s>0$, the process is a BM independent of $(B_u, 0\leq u\leq s)$.

(ii) The process is a BM

(iii) For any $c>0$, the process is a BM

(iv) The process $(X_t)_{t\geq 0}$ defined by for $t>0$ is a BM.

Proof (i) Define $X_t = B_{t+s} - B_s$. Then

$\Rightarrow$ independent Gaussian increments,

$(X_t)_{t\geq 0}$ has continuous paths $\Rightarrow$

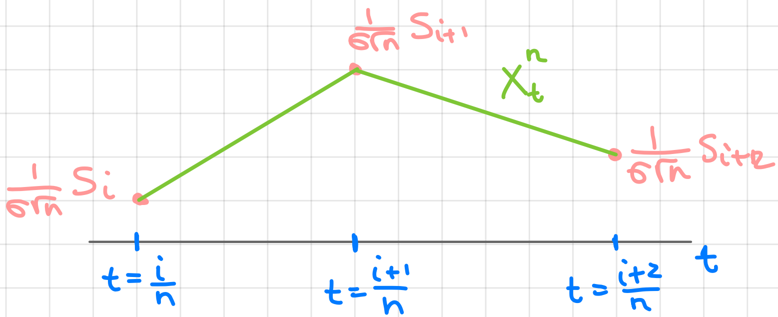
(iv) X_t is Gaussian, for $s<t$

Proof of $\lim_{t\to 0} X_t = 0$ is more technical, thus omitted.

Construction of BM

BM can be constructed as a limit of properly rescaled random walks.

Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. r.v.s, $E(\xi_i)=0$, $Var(\xi_i)=\sigma^2<\infty$. Denote and define



Theorem (Donsker)

Applying Donsker's theorem

Example Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d. r.v. $P(\xi_i = 1) = P(\xi_i = -1) = 0.5$

$E(\xi_i) = 0$, $Var(\xi_i) = 1$.

Denote $(S_m)_{m\geq 0}$ is a Markov chain.

From the first step analysis of MC we know that for any $-a < 0 < b$

If X^n_t is the process interpolating S_m, then $\forall n$

$P(X^n \text{ hits } -a \text{ before } b) =$

$\Rightarrow P(B \text{ hits } -a \text{ before } b) =$

$\Rightarrow (\tilde{\xi}_i)_{i=1}^{\infty}$, $E(\tilde{\xi}_i) = 0$, $Var(\tilde{\xi}_i) = 1$, $P(\tilde{S} \text{ hits } -a \text{ before } b) \approx \frac{b}{a+b}$