MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Brownian motion

Next: PK 8.1- 8.2

Week 9:

- homework 7 (due Friday, June 2)

Brownian motion. Definition

Def. **Brownian motion** with diffusion coefficient σ^2 is a continuous time stochastic process $(B_t)_{t\geq 0}$ satisfying

(i) $B(0)=0$, $B(t)$ is continuous as a function of t

(ii) For all $0\leq s<t<\infty$ $B(t)-B(s)$ is a Gaussian random variable with mean 0 and variance $\sigma^2(t-s)$

(iii) The increments of B are independent: if $0=t_0<t_1<\dots<t_n$ then $\{B(t_i)-B(t_{i-1})\}_{i=1}^{n}$ are independent (Gaussian) r.v.s.

$\sigma^2=1 \leftarrow$ standard BM

BM as a Gaussian process

Def. Stochastic process $(X_t)_{t\geq 0}$ is called a Gaussian process if for any $0 \leq t_1 < t_2 < \dots < t_n$

$(X_{t_1}, \dots, X_{t_n})$ is a Gaussian vector, or equivalently for any $c_1, \dots, c_n \in \mathbb{R}$

$c_1 X_{t_1} + \dots + c_n X_{t_n}$ is a Gaussian r.v.

Recall that the distribution of a Gaussian vector is uniquelly defined by its mean and covariance matrix.

Similarly, each Gaussian process is uniquely described by

$$\mu(t) = E(X_t) \quad \text{and} \quad \Gamma(s,t) = \mathrm{Cov}(X_s, X_t) \geq 0$$

covariance function

BM as a Gaussian process

Proposition BM is a Gaussian process with $(B_t)_{t\geq 0}$ $\mu(t) = 0$ and $\Gamma(s,t) = \min\{s,t\} = s\wedge t$

Proof. For any $0 \leq t_1 < t_2 < \cdots < t_n$, $B_{t_j} - B_{t_{j-1}}$ are indep. Gaussian, thus

$$\sum_{i=1}^{n} c_i B_{t_i} = \sum_{i=1}^{n} c_i \sum_{j=1}^{i} \left(B_{t_j} - B_{t_{j-1}}\right) = \sum_{j=1}^{n} \underbrace{\sum_{i=j}^{n} c_i}_{\alpha_j} \left(B_{t_j} - B_{t_{j-1}}\right)$$

is also Gaussian.

By definition $\mu(t) = E(X_t) = 0$. Let $s<t$.

Then
$$\begin{aligned}\Gamma(s,t) &= \mathrm{Cov}(B_s, B_t)\\ &= \mathrm{Cov}(B_s, B_s + (B_t - B_s))\\ &= \mathrm{Cov}(B_s, B_s) + \mathrm{Cov}(B_s, B_t - B_s)\\ &= s + 0 = s = \min\{s,t\}\end{aligned}$$

Some properties of BM

Proposition. Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i) For any $s>0$, the process $(\underbrace{B_{t+s}-B_s}_{X_t}, t\geq 0)$ is a BM independent of $(B_u, 0\leq u\leq s)$.

(ii) The process $(-B_t, t\geq 0)$ is a BM

(iii) For any $c>0$, the process $(c\,B_{\frac{t}{c^2}}, t\geq 0)$ is a BM

(iv) The process $(X_t)_{t\geq 0}$ defined by $X_0=0$, $X_t = t\,B_{\frac{1}{t}}$ for $t>0$ is a BM.

continuous on $(0,+\infty)$

Proof (i) Define $X_t = B_{t+s}-B_s$. Then $X_0=0$, $X_{t_2}-X_{t_1} = B_{s+t_2}-B_{s+t_1}$

$\Rightarrow$ independent Gaussian increments, $E(X_{t_2}-X_{t_1})=0$, $Var(X_{t_2}-X_{t_1}) = t_2-t_1$

$(X_t)_{t\geq 0}$ has continuous paths $\Rightarrow$ $(X_t)_{t\geq 0}$ is a BM

(iv) X_t is Gaussian, for $s<t$ $Cov(s\,B_{\frac{1}{s}}, t\,B_{\frac{1}{t}}) = st\min\{\frac{1}{s},\frac{1}{t}\} = s\wedge t$

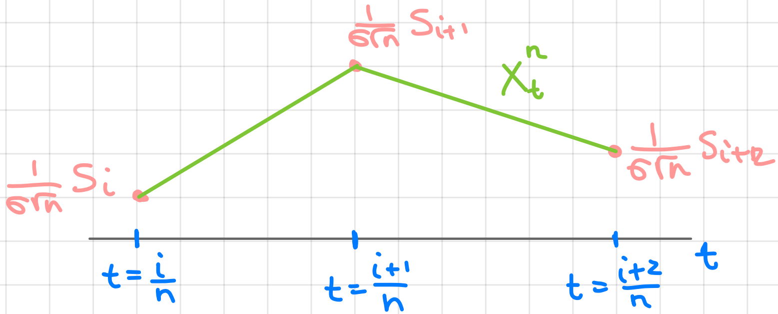
Proof of $\lim_{t\to 0} X_t = 0$ is more technical, thus omitted.

Construction of BM

BM can be constructed as a limit of properly rescaled random walks.

Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. r.v.s, $E(\xi_i)=0$, $Var(\xi_i)=\sigma^2<\infty$. Denote $S_m=\sum_{i=1}^{m}\xi_i$ and define

$$X_t^n=\frac{1}{\sigma\sqrt{n}}\left(S_{[nt]}+(nt-[nt])\,\xi_{[nt]+1}\right)$$



Theorem (Donsker) $(X_t^n)_{t\geq 0}$ converges in distribution to the standard BM

Applying Donsker's theorem

Example Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d. r.v. $P(\xi_i=1)=P(\xi_i=-1)=0.5$

$E(\xi_i)=0$, $Var(\xi_i)=1$.

Denote $S_m := \sum_{i=1}^{m} \xi_i$, $S_0=0$ $(S_m)_{m\geq 0}$ is a Markov chain.

From the first step analysis of MC we know that for any $-a<0<b$ $\quad P(S_m \text{ hits } -a \text{ before } b) = \frac{b}{a+b}$

If X^n_t is the process interpolating S_m, then $\forall n$

$$P(X^n \text{ hits } -a \text{ before } b) = P(S \text{ hits } -\sqrt{n}\,a \text{ before } \sqrt{n}\,b)$$

$$= \frac{\sqrt{n}\,b}{\sqrt{n}\,a+\sqrt{n}\,b} = \frac{b}{a+b}$$

$\Rightarrow P(B \text{ hits } -a \text{ before } b) = \frac{b}{a+b}$

$\Rightarrow (\tilde{\xi}_i)_{i=1}^{\infty}$, $E(\tilde{\xi}_i)=0$, $Var(\tilde{\xi}_i)=1$, $P(\tilde{S} \text{ hits } -a \text{ before } b) \approx \frac{b}{a+b}$