MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Brownian motion

Next: PK 8.3- 8.4

Week 10:

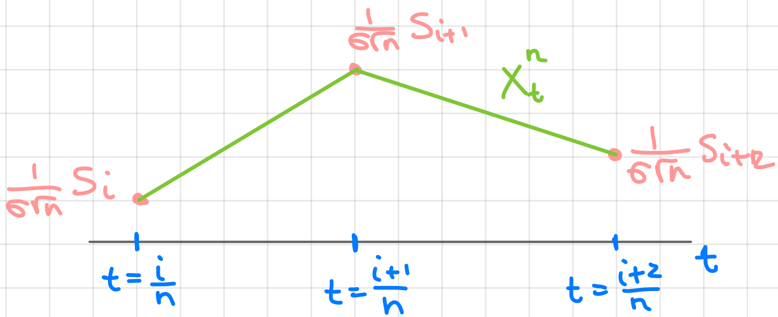
- homework 8 (due Friday, June 9)

Construction of BM

BM can be constructed as a limit of properly rescaled random walks.

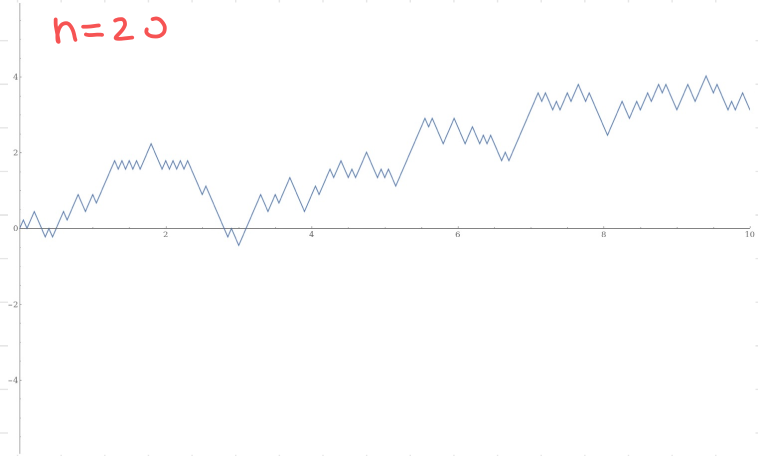
Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. r.v.s, $E(\xi_i)=0$, $Var(\xi_i)=\sigma^2<\infty$. Denote $S_m=\sum_{i=1}^{m}\xi_i$ and define

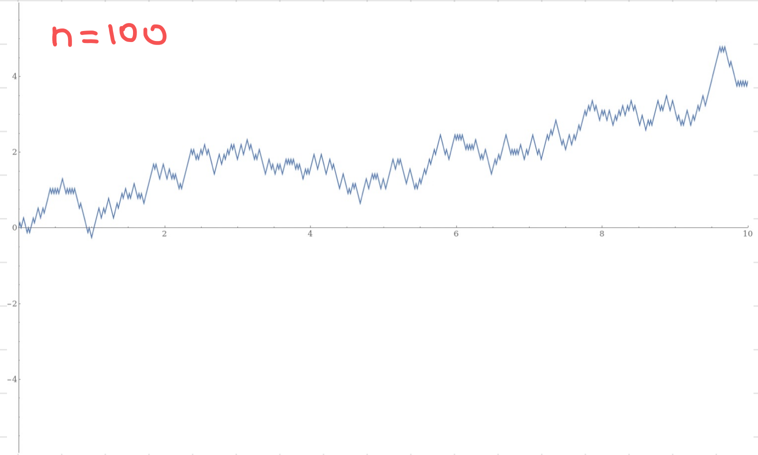
$$X_t^n=\frac{1}{\sigma\sqrt{n}}\left(S_{[nt]}+(nt-[nt])\,\xi_{[nt]+1}\right)$$

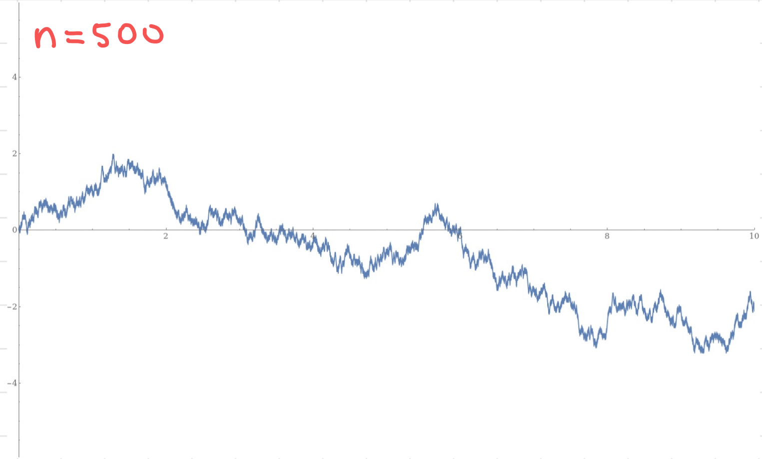


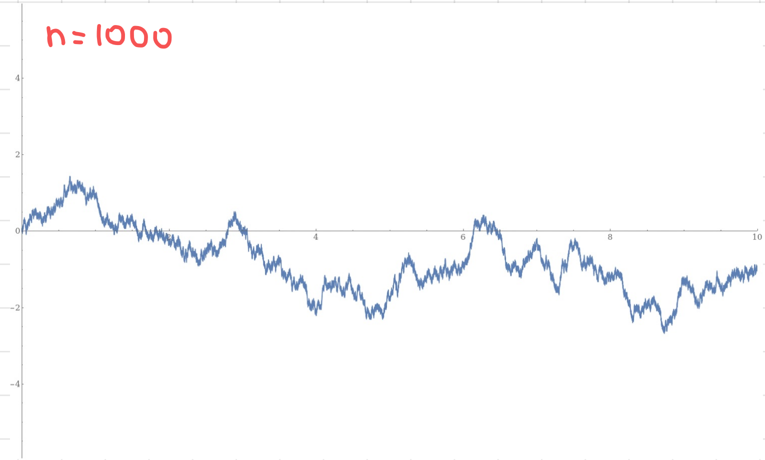
Theorem (Donsker) $(X_t^n)_{t\geq 0}$ converges in distribution to the standard BM

Approximating a BM with random walks X_t^n









Applying Donsker's theorem

Example Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d. r.v. $P(\xi_i = 1) = P(\xi_i = -1) = 0.5$

$E(\xi_i) = 0$, $Var(\xi_i) = 1$.

Denote $S_m := \sum_{i=1}^{m} \xi_i$, $S_0 = 0$ $(S_m)_{m \geq 0}$ is a Markov chain.

From the first step analysis of MC we know that for any $-a < 0 < b$ $P(S_m \text{ hits } -a \text{ before } b) = \frac{b}{a+b}$

If X_t^n is the process interpolating S_m, then $\forall n$

$$P(X^n \text{ hits } -a \text{ before } b) = P(S \text{ hits } -\sqrt{n}\, a \text{ before } \sqrt{n}\, b)$$

$$= \frac{\sqrt{n}\, b}{\sqrt{n}\, a + \sqrt{n}\, b} = \frac{b}{a+b}$$

$$\Rightarrow P(B \text{ hits } -a \text{ before } b) = \frac{b}{a+b}$$

BM as a martingale

Let $(X_t)_{t\geq 0}$ be a continuous time stochastic process. We say that $(X_t)_{t\geq 0}$ is a martingale if $E(|X_t|)<\infty$ $\forall t\geq 0$ and

$$E\left(X_t \mid \{ X_u ,\ 0\leq u\leq s\} \right) = X_s \qquad s<t$$

Proposition Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i) $(B_t)_{t\geq 0}$ is a martingale

(ii) $(B_t^2 - t)_{t\geq 0}$ is a martingale (w.r.t. $(B_t)_{t\geq 0}$)

"Proof": $E(B_t \mid \{B_u , 0\leq u\leq s\}) = E(B_t - B_s + B_s \mid \{B_u, 0\leq u\leq s\}) = B_s + 0 = B_s$

$$E(B_t^2 - t \mid \{B_u, 0\leq u\leq s\}) = E\left(B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2 \mid \{ B_u , 0\leq u\leq s\}\right) - t$$

$$= B_s^2 + 0 + t - s - t = B_s^2 - s$$

Thm (Lévy) Let $(X_t)_{t\geq 0}$ be a continuous martingale such that $(X_t^2 - t)_{t\geq 0}$ is a martingale. Then (X_t) is a sB

Stopping times and the strong Markov property (lec. 5)

<u>Def</u> (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a stopping time if the event

$$\{T \leq t\}$$

can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

<u>Examples</u>: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = x\}$ is a stopping time

2. $\sup\{t \geq 0 : X_t = x\}$ is not a stopping time

Stopping times and the strong Markov property (lec 5)

Theorem (no proof)

Let $(X_t)_{t\geq 0}$ be a Markov process, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T = x$,

$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

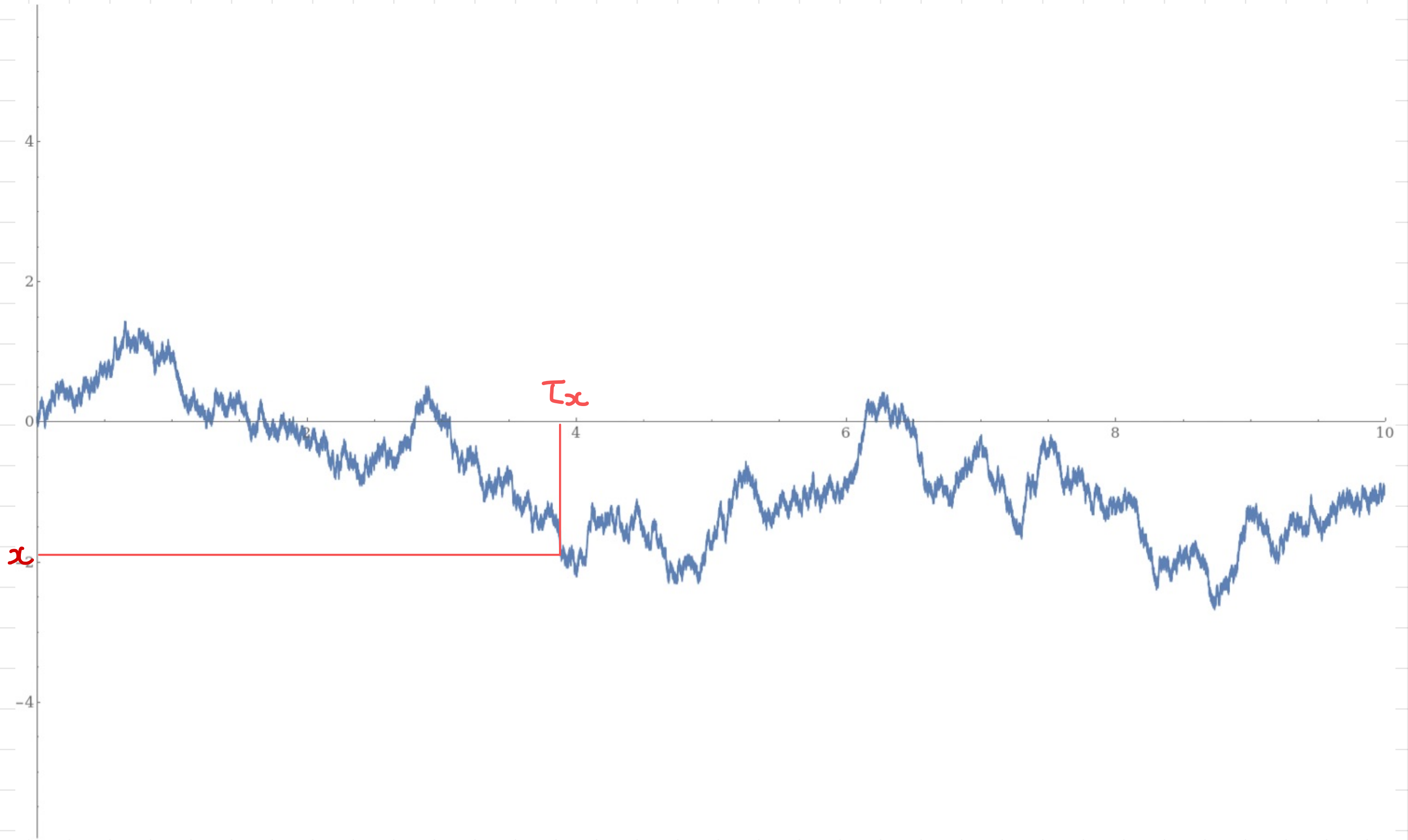
(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from x

Example $(B_t)_{t\geq 0}$ is Markov. For any $x\in\mathbb{R}$ define

$$\tau_x = \min\{t : B_t = x\}.$$ Then

- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is a BM starting from 0
- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is independent of $\{B_s,\ 0\leq s\leq \tau_x\}$

(independent of what B was doing before it hit x)



4
2
0
-4
4
6
8
10
τx
x

Reflection principle

Thm. Let $(B_t)_{t\geq 0}$ be a standard BM. Then for any $t \geq 0$ and $x > 0$

$$\cancel{(S_t)_{t\geq 0} \overset{(d)}{=} (|B_t|)_{t\geq 0}}$$

$$P\Big(\overbrace{\max_{0\leq u\leq t} B_u}^{S_t} \geqslant x\Big) = P\big(|B_t| \geqslant x\big) = 2\cdot P(B_t \geq x)$$

Proof. Let $\tau_x = \min\{t : B_t = x\}$. Note that τ_x is a stopping time and is uniquely determined by $\{B_u, 0\leq u\leq \tau_x\}$. From the definition of τ_x, $\max_{0\leq u\leq t} B_u \geq x \Leftrightarrow \tau_x \leq t$. Then

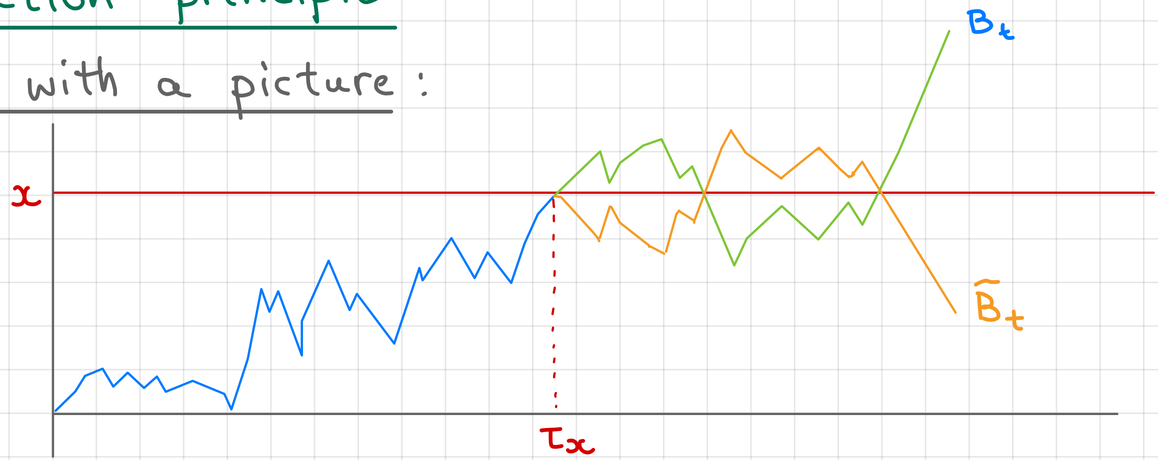
$$P\big(\max_{0\leq u\leq t} B_u \geq x\big) = P\big(\tau_x \leq t,\ B_t > x\big) + P\big(\tau_x \leq t,\ B_t < x\big)$$

$$= P\big(B_t > x\big) + \underbrace{P\big(\tau_x \leq t,\ B_t < x\big)}_{P(B_t > x)}$$

$$= 2\,P\big(B_t > x\big) = P\big(|B_t| > x\big)$$

Reflection principle

Proof with a picture:



If $(B_t)_{t\geq 0}$ is a BM, then $(\tilde{B}_t)_{t\geq 0}$ is a BM, where

$$\tilde{B}_t = \begin{cases} B_t , & t \leq \tau_x \\ B_{\tau_x} - (B_t - B_{\tau_x}), & t > \tau_x \end{cases}$$

$\Rightarrow$ to each sample path with $\max\limits_{0\leq u\leq t} B_u > x$ and $B_t > x$ we associate a unique path with $\max\limits_{0\leq u\leq t} \tilde{B}_u > x$ and $\tilde{B}_t < x$, so

$$P(\tau_x \leq t, B_t < x) = P(\tau_x \leq t, B_t > x) = P(B_t > x)$$

$\blacksquare$

Application of the RP: distribution of the hitting time τ_x

By definition, $\tau_x \le t \iff \max\limits_{0\le u\le t} B_t \ge x$, so

$$P(\tau_x \le t) = P\left(\max_{0\le u\le t} B_u \ge x\right) = 2\,P(B_t \ge x)$$

$$= 2\cdot\frac{1}{\sqrt{2\pi t}}\int_x^{+\infty} e^{-\frac{u^2}{2t}}\,du \qquad \left\{\begin{array}{l} u = \sqrt{t}\,v \\ du = \sqrt{t}\,dv \end{array}\right.$$

$$= 2\cdot\frac{1}{\sqrt{2\pi}}\int_{x/\sqrt{t}}^{+\infty} e^{-\frac{v^2}{2}}\,dv$$

$\Rightarrow$ p.d.f. of τ_x $\quad f_{\tau_x}(t) = \sqrt{\frac{2}{\pi}}\, e^{-\frac{x^2}{2t}}\, x\,\frac{1}{2}\,t^{-3/2} = \frac{1}{\sqrt{2\pi}}\,x\,t^{-3/2}\,e^{-\frac{x^2}{2t}}$

<u>Thm</u>. $\quad F_{\tau_x}(t) = \sqrt{\frac{2}{\pi}}\int_{x/\sqrt{t}}^{\infty} e^{-\frac{v^2}{2}}\,dv\,,$

$$f_{\tau_x}(t) = \frac{x}{\sqrt{2\pi}}\,t^{-3/2}\,e^{-\frac{x^2}{2t}}$$