MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Brownian motion

Next: PK 8.3- 8.4

Week 10:

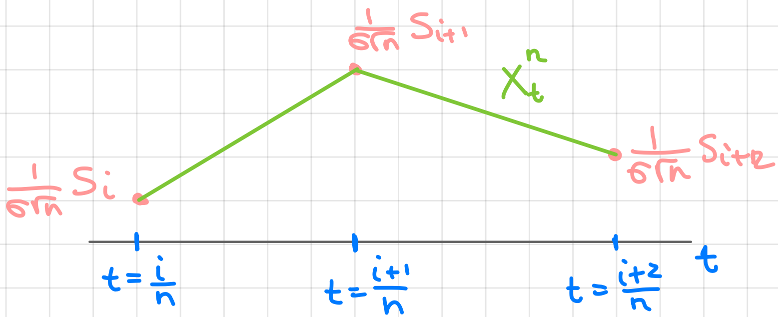
- homework 8 (due Friday, June 9)

Construction of BM

BM can be constructed as a limit of properly rescaled random walks.

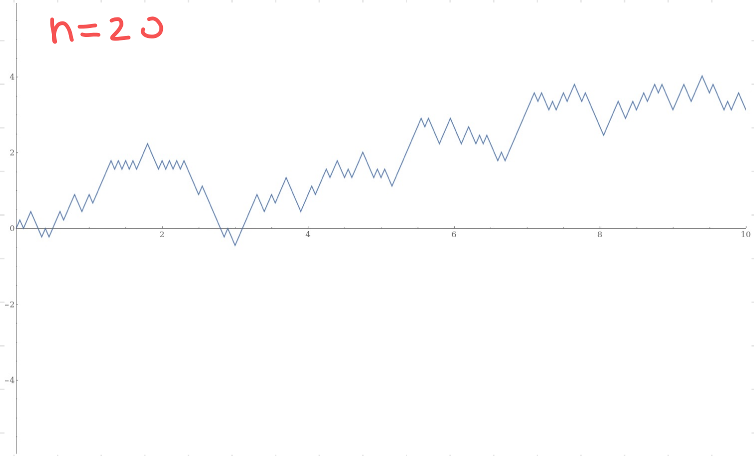
Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. r.v.s, $E(\xi_i)=0$, $Var(\xi_i)=\sigma^2<\infty$. Denote $S_m=\sum_{i=1}^{m}\xi_i$ and define

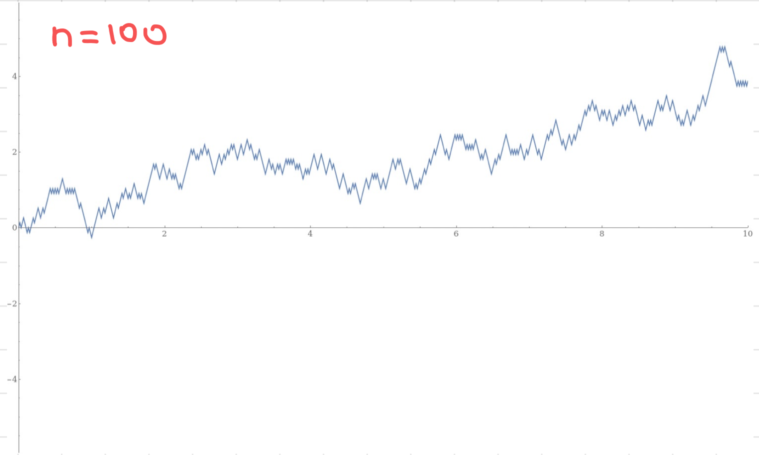
$$X_t^n=\frac{1}{\sigma\sqrt{n}}\left(S_{[nt]}+(nt-[nt])\,\xi_{[nt]+1}\right)$$

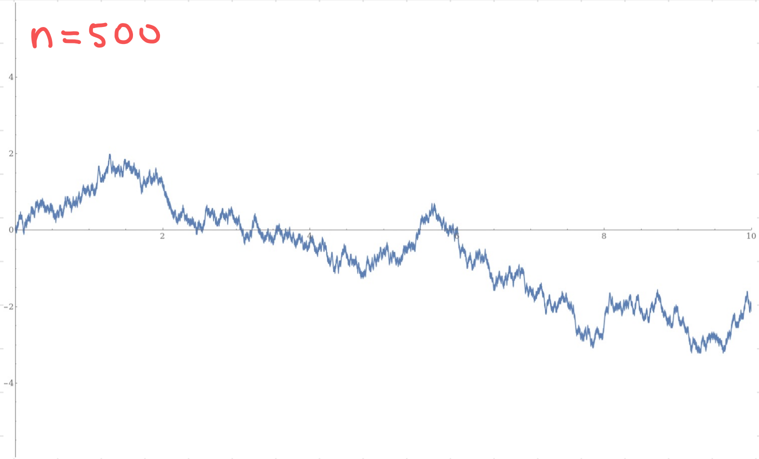


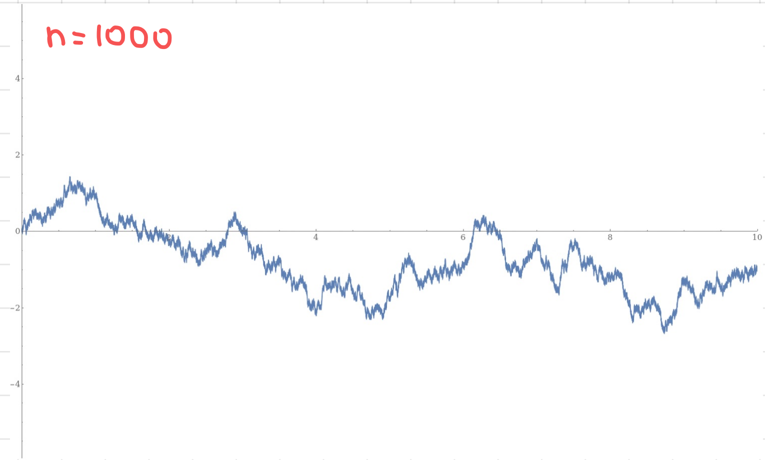
Theorem (Donsker) $(X_t^n)_{t\geq 0}$ converges in distribution to the standard BM

Approximating a BM with random walks X^n_t









Applying Donsker's theorem

Example Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d. r.v. $P(\xi_i=1)=P(\xi_i=-1)=0.5$

$E(\xi_i)=0$, $Var(\xi_i)=1$.

Denote $S_m := \sum_{i=1}^{m} \xi_i$, $S_0=0$ $(S_m)_{m\geq 0}$ is a Markov chain.

From the first step analysis of MC we know that for any $-a<0<b$ $P(S_m \text{ hits } -a \text{ before } b) = \frac{b}{a+b}$

If X_t^n is the process interpolating S_m, then $\forall n$

$$P(X^n \text{ hits } -a \text{ before } b) = P(S \text{ hits } -\sqrt{n}a \text{ before } \sqrt{n}b)$$

$$= \frac{\sqrt{n}b}{\sqrt{n}a+\sqrt{n}b} = \frac{b}{a+b}$$

$\Rightarrow P(B \text{ hits } -a \text{ before } b) = \frac{b}{a+b}$

$\Rightarrow (\tilde{\xi}_i)_{i=1}^{\infty}$, $E(\tilde{\xi}_i)=0$, $Var(\tilde{\xi}_i)=1$, $P(\tilde{S} \text{ hits } -a \text{ before } b) \approx \frac{b}{a+b}$

BM as a martingale

Let $(X_t)_{t\geq 0}$ be a continuous time stochastic process. We say that $(X_t)_{t\geq 0}$ is a martingale if $E(|X_t|) < \infty$ $\forall t \geq 0$ and

<u>Proposition</u> Let $(B_t)_{t\geq 0}$ be a standard BM. Then

(i)

(ii)

"Proof": $E(B_t \mid \{B_u, 0 \leq u \leq s\}) =$

$E(B_t^2 - t \mid \{B_u, 0 \leq u \leq s\}) =$

$=$

<u>Thm</u> (Lévy) Let $(X_t)_{t\geq 0}$ be a continuous martingale such that $(X_t^2 - t)_{t\geq 0}$ is a martingale.

Stopping times and the strong Markov property (lec. 5)

Def (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a stopping time if the event

$$\{T \leq t\}$$

can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = x\}$ is a stopping time

2. $\sup\{t \geq 0 : X_t = x\}$ is not a stopping time

Stopping times and the strong Markov property (lec 5)

Theorem (no proof)

Let $(X_t)_{t\geq 0}$ be a Markov process, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T = x$,

$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

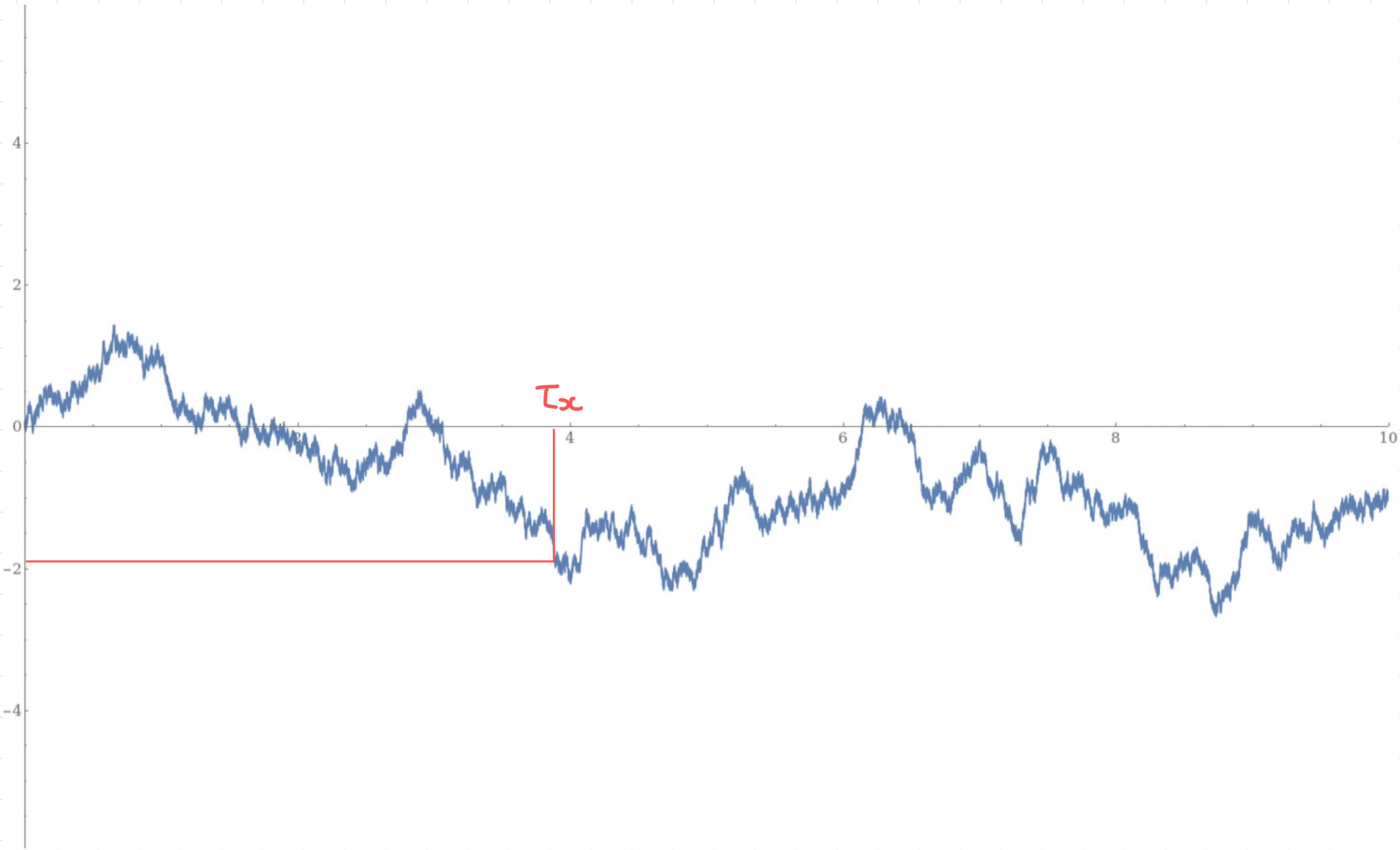
(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from x

Example $(B_t)_{t\geq 0}$ is Markov. For any $x\in\mathbb{R}$ define

$\tau_x = \min\{t : B_t = x\}$. Then

- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is a BM starting from x
- $(B_{t+\tau_x} - B_{\tau_x})_{t\geq 0}$ is independent of $\{B_s,\ 0\leq s\leq \tau_x\}$

(independent of what B was doing before it hit x)



4
2
0
-2
-4
4
6
8
10
τ_x

Reflection principle

<u>Thm</u>. Let $(B_t)_{t\geq 0}$ be a standard BM. Then for any $t \geq 0$ and $x > 0$

<u>Proof</u>. Let $\tau_x = \min\{t : B_t = x\}$. Note that τ_x is a stopping time and is uniquely determined by $\{B_u, 0 \leq u \leq \tau_x\}$

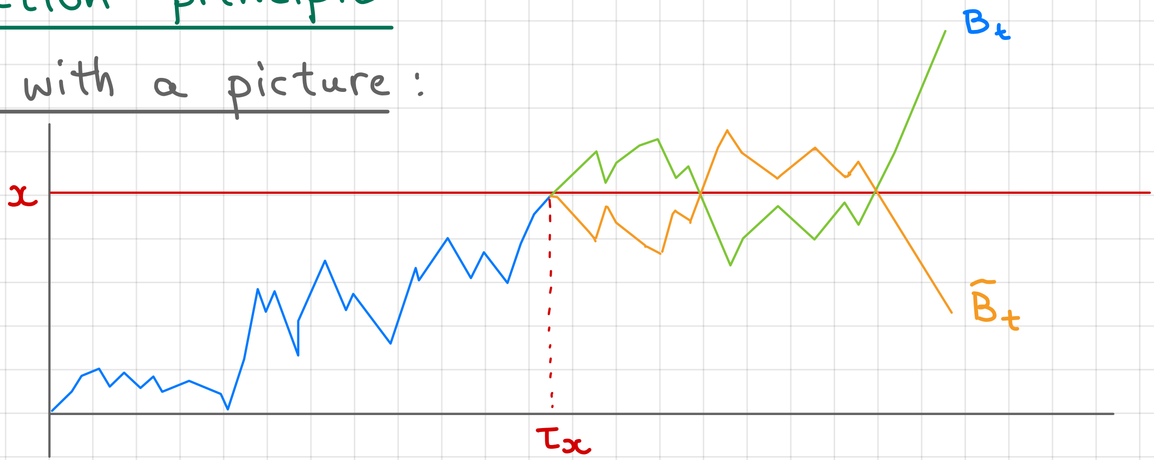
From the definition of τ_x, . Then

$$P\left(\max_{0\leq u\leq t} B_u \geq x,\ B_t < x\right) =$$

Now $P\left(\max_{0\leq u\leq t} B_u \geq x\right) =$

Reflection principle

Proof with a picture:



If $(B_t)_{t\geq 0}$ is a BM, then $(\tilde{B}_t)_{t\geq 0}$ is a BM, where

$$\tilde{B}_t = \begin{cases} B_t \,, & t \leq \tau_x \\ B_{\tau_x} - (B_t - B_{\tau_x}) \,, & t > \tau_x \end{cases}$$

$\Rightarrow$ to each sample path with $\max\limits_{0\leq u\leq t} B_u > x$ and $B_t > x$ we associate a unique path with $\max\limits_{0\leq u\leq t} \tilde{B}_u > x$ and $\tilde{B}_t < x$, so

$$P\left(\max_{0\leq u\leq t} B_u \geq x,\, B_t < x\right) = P\left(B_t > x\right) \Rightarrow P\left(\max_{0\leq u\leq t} B_u \geq x\right) = 2P\left(B_t \geq x\right) \blacksquare$$

Application of the RP: distribution of the hitting time τ_x

By definition, $\tau_x \leq t \iff \max_{0 \leq u \leq t} B_t \geq x$, so

$$P(\tau_x \leq t) =$$

$$=$$

$$=$$

$\Rightarrow$ p.d.f. of τ_x $\quad f_{\tau_x}(t) =$

<u>Thm</u>. $F_{\tau_x}(t) = \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{t}}^{\infty} e^{-\frac{v^2}{2}} dv$,

$$f_{\tau_x}(t) = \frac{x}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{x^2}{2t}}$$

Zeros of BM

Denote by $\theta(t,t+s)$ the probability that $B_u = 0$ on $(t,t+s)$

$$\theta(t,t+s) :=$$

<u>Thm</u>. For any $t, s > 0$

$$\theta(t,t+s) =$$

<u>Proof</u> Compute $P(\,B_u = 0$ for some $u \in (t,t+s])$ by conditioning on the value of B_t.

$$\theta(t,t+s) =$$

(*)

Define $\tilde{B}_u = B_{t+u} - B_t$. Then

$$P(\,B_u = 0 \text{ on } (t,t+s] \mid B_t = x) =$$

(**)

Zeros of BM

Plugging (**) into (*) gives

$$\Theta(t, t+s) = \int_{-\infty}^{+\infty} P(B_u = x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$= \int_{0}^{+\infty} P(B_u = x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$+ \int_{0}^{\infty} P(B_u = -x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$=$$

Finally, $P(B_u = x > 0 \text{ for some } u \in (0,s]) =$

$$(*) = \int_0^{\infty} \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}} \left(\int_0^s \frac{x}{\sqrt{2\pi}} y^{-3/2} e^{-\frac{x^2}{2y}} dy \right) dx =$$

Zeros of BM

$$\int_0^\infty x e^{-\frac{x^2}{2}\left(\frac{1}{t}+\frac{1}{y}\right)} dx =$$

$\Rightarrow (*) =$

Now use the change of variable $z=\sqrt{\frac{y}{t}}$, $dy = 2tdz$

$$(*) = \frac{\sqrt{t}}{\pi}\int_0^{\sqrt{s/t}} \frac{1}{t(1+z^2)\sqrt{t}\,z}\cdot 2t\,dz = \frac{2}{\pi}\int_0^{\sqrt{s/t}} \frac{1}{1+z^2}dz = \frac{2}{\pi}\arctan\left(\sqrt{\frac{s}{t}}\right)$$

$$= \frac{2}{\pi}\arccos\left(\sqrt{\frac{t}{s+t}}\right)$$

exercise

$\blacksquare$

Remark Let $T_0 := \inf\{t>0 : B_t = 0\}$. Then $P(T_0=0)=1$

There is a sequense of zeros of $B_t(\omega)$ converging to 0.

To understand the structure of the set of zeros → Cantor set

Behavior of BM as $t \to \infty$

Thm. Let $(B_t)_{t\geq 0}$ be a (standard) BM. Then

$$P\left(\sup_{t\geq 0} B_t = +\infty, \inf_{t\geq 0} B_t = -\infty\right) = 1$$

(BM "oscilates with increasing amplitude")

Proof. Denote $Z = \sup_{t\geq 0} B_t$. Then for any $c > 0$

$$cZ =$$

By property (iii), cB_{t/c^2} is a standard BM, so cZ has the same distribution as $Z \Rightarrow P(Z=0)=p, P(Z=\infty)= 1-p$

$p=P(Z=0)$

$\Rightarrow P(Z=0)=0, P(Z=\infty)=1$. Similarly for $\inf_{t\geq 0} B_t$ ∎