MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Brownian motion

Next: PK 8.3- 8.4

Week 10:

- homework 8 (due Friday, June 9)

Reflection principle

Thm. Let $(B_t)_{t\geq 0}$ be a standard BM. Then for any $t \geq 0$ and $x > 0$

$$\cancel{(S_t)_{t\geq 0} \overset{(d)}{=} (|B_t|)_{t \geq 0}}$$

$$P\Big(\overbrace{\max_{0\leq u\leq t} B_u}^{S_t} \geq x\Big) = P\big(|B_t| \geq x\big) = 2\cdot P\big(B_t \geq x\big)$$

Let $\tau_x = \min\{t : B_t = x\}$.

Thm. $F_{\tau_x}(t) = \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{t}}^{\infty} e^{-\frac{v^2}{2}} dv$, $\qquad f_{\tau_x}(t) = \frac{x}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{x^2}{2t}}$

Zeros of BM

Denote by $\theta(t, t+s)$ the probability that $B_u = 0$ on $(t, t+s)$

$$\theta(t, t+s) := P\big(B_u = 0 \text{ for some } u \in (t, t+s)\big)$$

Thm. For any $t, s > 0$

$$\theta(t, t+s) = \frac{2}{\pi} \arccos \sqrt{\frac{t}{t+s}}$$

Proof Compute $P(B_u = 0 \text{ for some } u \in (t, t+s])$ by conditioning on the value of B_t.

$$\theta(t, t+s) = \int_{-\infty}^{+\infty} P(B_u = 0 \text{ for some } u \in (t, t+s] \mid B_t = x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

(*)

Define $\tilde{B}_u = B_{t+u} - B_t$. Then

$$P(B_u = 0 \text{ on } (t, t+s] \mid B_t = x) = P(\tilde{B}_u = -x \text{ on } (0, s] \mid B_t = x)$$

(**)

$$= P(\tilde{B}_u = -x \text{ on } (0, s]) = P(\tilde{B}_u = x \text{ on } (0, s])$$

Zeros of BM

Plugging (**) into (*) gives

$$\Theta(t,t+s) = \int_{-\infty}^{+\infty} P(B_u = x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$= \int_{0}^{+\infty} P(B_u = x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$+ \int_{0}^{\infty} P(B_u = -x \text{ for some } u \in (0,s]) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

$$= \sqrt{\frac{2}{\pi t}} \int_{0}^{\infty} P(B_u = x \text{ for some } u \in (0,s]) \, e^{-\frac{x^2}{2t}} dx$$

Finally, $P(B_u = x > 0 \text{ for some } u \in (0,s]) = P(\max_{0 \le u \le s} B_u \ge x) = P(\tau_x \le s)$

$$(*) = \int_{0}^{\infty} \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}} \left(\int_{0}^{s} \frac{x}{\sqrt{2\pi}} y^{-3/2} e^{-\frac{x^2}{2y}} dy \right) dx = \frac{1}{\pi\sqrt{t}} \int_{0}^{s} \int_{0}^{\infty} x \, e^{-\frac{x^2}{2}\left(\frac{1}{t}+\frac{1}{y}\right)} dx \, y^{-3/2} dy$$

Zeros of BM

$$\frac{x^2}{2} = w$$

$$\int_0^\infty x e^{-\frac{x^2}{2}\left(\frac{1}{t}+\frac{1}{y}\right)} dx = \int_0^\infty e^{-w\left(\frac{1}{t}+\frac{1}{y}\right)} dw = \frac{1}{\frac{1}{t}+\frac{1}{y}} = \frac{ty}{t+y}$$

$$y = tz^2 \qquad \sqrt{y} = \sqrt{t}\, z \qquad dy = 2tz\, dz$$

$$\Rightarrow (*) = \frac{1}{\pi}\int_0^s \frac{\sqrt{t}}{t+y}\, y^{-1/2}\, dy = \frac{1}{\pi}\int_0^{\sqrt{s/t}} \frac{\cancel{\sqrt{t}}}{t+tz^2}\, \frac{1}{\cancel{\sqrt{t}}\, \cancel{z}}\, 2t\cancel{z}\, dz$$

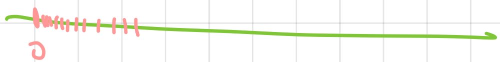
Now use the change of variable $z = \sqrt{\frac{y}{t}}$, $dy = 2t\, dz$

$$(*) = \frac{\sqrt{t}}{\pi}\int_0^{\sqrt{s/t}} \frac{1}{t(1+z^2)\sqrt{t}\, z}\cdot 2zt\, dz = \frac{2}{\pi}\int_0^{\sqrt{s/t}} \frac{1}{1+z^2}\, dz = \frac{2}{\pi}\arctan\left(\sqrt{\frac{s}{t}}\right)$$

$$= \frac{2}{\pi}\arccos\left(\sqrt{\frac{t}{s+t}}\right)$$

exercise

$(0, \varepsilon)$



$\blacksquare$

Remark Let $T_0 := \inf\{t > 0 : B_t = 0\}$. Then $P(T_0 = 0) = 1$

There is a sequense of zeros of $B_t(\omega)$ converging to 0.

To understand the structure of the set of zeros $\rightarrow$ Cantor set

Behavior of BM as $t \to \infty$

Thm. Let $(B_t)_{t\geq 0}$ be a (standard) BM. Then

$$P\left(\sup_{t\geq 0} B_t = +\infty, \inf_{t\geq 0} B_t = -\infty\right) = 1$$

(BM "oscilates with increasing amplitude")

Proof. Denote $Z = \sup_{t\geq 0} B_t$. Then for any $c > 0$

$$cZ = \sup_{t\geq 0} cB_t = \sup_{t\geq 0} cB_{t/c^2} \sim Z$$

By property (iii), cB_{t/c^2} is a standard BM, so cZ has the same distribution as $Z \Rightarrow P(Z=0)=p, P(Z=\infty)=1-p$

$$p = P(Z=0) \leq P\left(B_1 \leq 0 \text{ and } \sup_{t\geq 0} B_{t+1} - B_1 = 0\right) = \frac{1}{2}\cdot p$$

$\Rightarrow \underline{P(Z=0)=0}$, $P(Z=\infty)=1$. Similarly for $\inf_{t\geq 0} B_t$ ∎

Sample paths of $(B_t)_t$ are not differentiable

Thm. $P(\ B_t$ is not differentiable at zero $)=1$

Proof. $P(\ \sup_{t\geq 0} B_t = \infty\ ,\ \inf_{t\geq 0} B_t = -\infty\ )=1.$ $(\star)$

Consider $\tilde{B}_t = t B_{1/t}$. $(\tilde{B}_t)_{t\geq 0}$ is a BM (by property (iv))

By $(\star)$, for any $\varepsilon>0$ $\exists\ t<\varepsilon,\ s<\varepsilon$ such that

$\tilde{B}_t > 0,\ \tilde{B}_s < 0 \Rightarrow$ only differentiable if $\tilde{B}_0' = 0$

But if $\tilde{B}_0' = 0$, then

for some $t>0$ and all $0<s<t$,

which imples that for all $0<s<t$, which

contradicts to $(\star)$ ∎

Thm $P(\ (B_t)_{t\geq 0}$ is nowhere differentiable $)=1$

Reflected BM

Def. Let $(B_t)_{t\geq 0}$ be a standard BM. The stochastic process

$$R_t := |B_t| = \begin{cases} B_t, & \text{if } B(t) \geq 0 \\ -B_t, & \text{if } B(t) < 0 \end{cases}$$

is called reflected BM.

Think of a movement in the vicinity of a boundary.

Moments: $E(R_t) = \int_{-\infty}^{+\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = 2 \int_0^{\infty} x \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = \sqrt{\frac{2t}{\pi}}$

$$Var(R_t) = E(B_t^2) - (E(|B_t|))^2 = t - \left(\sqrt{\frac{2t}{\pi}}\right)^2 = \left(1 - \frac{2}{\pi}\right)t$$

Transition density: $P(R_t \leq y \mid R_0 = x) = P(-y \leq B_t \leq y \mid B_0 = x)$

$$= \quad \Rightarrow p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}} \right)$$

Thm (Lévy) Let $M_t = \max_{0 \leq u \leq t} B_u$. Then $(M_t - B_t)_{t \geq 0}$ is a reflected BM.

Reflected BM