MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Introduction. Birth processes

Next: PK 6.2-6.3

Week 1:

- join Piazza

Continuous Time Markov Chains

<u>Def</u> (Continuous-time Markov chain)

Let $(X_t)_{t\geq 0} = (X_t : 0 \leq t < \infty)$ be a continuous time process taking values in $\mathbb{Z}_+$. $(X_t)_{t\geq 0}$ is called Markov chain if for any $n \in \mathbb{N}$, $0 \leq t_0 < t_1 < \cdots < t_{n-1} < s$, $t > 0$, $i_0, i_1, \ldots, i_{n-1}, i, j \in \mathbb{Z}_+$

$$P(X_{s+t} = j \mid X_{t_0} = i_0, X_{t_1} = i_1, \ldots, X_{t_{n-1}} = i_{n-1}, X_s = i)$$
$$= P(X_{s+t} = j \mid X_s = i)$$

Transition probability function

One way of describing a continuous time MC is by using the transition probability functions.

Def. Let $(X_t)_{t\geq 0}$ be a MC. We call

$$P(X_{s+t}=j \mid X_s=i) \qquad i,j\in\{0,1,\dots\}, \quad s\geq 0, \ t>0$$

the transition probability function for $(X_t)_{t\geq 0}$.

If $P(X_{s+t}=j \mid X_s=i)$ does not depend on s, we say that $(X_t)_{t\geq 0}$ has stationary transition probabilities and we define

$$P_{ij}(t) := P(X_{s+t}=j \mid X_s=i) = P(X_t=j \mid X_0=i)$$

[compare with n-step transition probabilities]

Characterization of the Poisson process

Experiment: count events occurring along $[0, +\infty)$ $\left\{\begin{array}{l}\text{time} \\ \text{or 1-D space}\end{array}\right.$

Denote by $N((a,b])$ the number of events that occur on $(a,b]$.

Assumptions:

1. Number of events happening in disjoint intervals are independent.
2. For any $t \geq 0$ and $h > 0$, the distribution of $N((t,t+h])$ does not depend on t (only on h, the length of the interval)
3. There exists $\lambda > 0$ s.t. $P(N((t,t+h]) \geq 1) = \lambda h + o(h)$ as $h \to 0$
   (rare events)
4. Simultaneous events are not possible: $P(N((t,t+h]) \geq 2) = o(h), h \to 0$

$X_t := N((0,t])$ is a Poisson process of rate λ

Transition probabilities of the Poisson process

Let $(X_t)_{t\geq 0}$ be the Poisson process.

Define the transition probability functions

$$P_{ij}(h) := P(X_{t+h} = j \mid X_t = i), \quad i,j \in \{0,1,2,\dots\}, \quad t \geq 0, \; h > 0$$

What are the infinitesimal (small h) transition probability functions for $(X_t)_{t\geq 0}$? As $h \to 0$,

$$P_{ii}(h) = P(X_{t+h} = i \mid X_t = i)$$
$$= P(X_{t+h} - X_t = 0 \mid X_t = i) = P(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h)$$

$$P_{i,i+1}(h) = P(X_{t+h} = i+1 \mid X_t = i) = P(X_{t+h} - X_t = 1) = \lambda h + o(h)$$

$$\sum_{j \notin \{i, i+1\}} P_{ij}(h) = o(h)$$

Poisson process and transition probabilities

To sum up: $(X_t)_{t\geq 0}$ is a MC with (infinitesimal) transition probabilities satisfying

$$P_{ii}(h) = 1 - \lambda h + o(h)$$

$$P_{i,i+1}(h) = \lambda h + o(h)$$

$$\sum_{j \notin \{i,i+1\}} P_{i,j}(h) = o(h)$$

What if we allow $P_{ij}(h)$ depend on i?

$\hookrightarrow$ birth and death processes

Pure birth processes

Def Let $(\lambda_k)_{k\geq 0}$ be a sequence of positive numbers. We define a pure birth process as a Markov process $(X_t)_{t\geq 0}$ whose stationary transition probabilities satisfy

1. $P_{k,k+1}(h) = \lambda_k h + o(h)$
2. $P_{k,k}(h) = 1 - \lambda_k h + o(h)$
3. $P_{k,j}(h) = 0 \quad$ for $j < k$
4. $X_0 = 0$

Related model. Yule process : $\lambda_k = \beta k$ for some $\beta > 0$.

Describes the growth of a population

- birth rate is proportional to the size of the population

Birth processes and related differential equations

Now define $P_n(t) = P(X_t = n)$. For small $h > 0$

$$P_n(t+h) = P(X_{t+h} = n) = \sum_{k=0}^{n} P(X_{t+h} = n \mid X_t = k)\, P(X_t = k)$$

$$= \sum_{k=0}^{n} P_{k,n}(h) \cdot P(X_t = k)$$

$$= P_{n,n}(h) \cdot P_n(t) + P_{n-1,n}(h) \cdot P_{n-1}(t) + \sum_{k=0}^{n-2} P_{k,n}(h) P_k(t)$$

$$= (1 - \lambda_n h) P_n(t) + \lambda_{n-1} h \cdot P_{n-1}(t) + o(h)$$

$$= P_n(t) - \lambda_n h P_n(t) + \lambda_{n-1} h \cdot P_{n-1}(t) + o(h)$$

$$P_n(t+h) - P_n(t) = -\lambda_n h P_n(t) + \lambda_{n-1} h P_{n-1}(t) + o(h)$$

$$P_n'(t) = \lim_{h \to 0} \frac{P_n(t+h) - P_n(t)}{h} = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t)$$

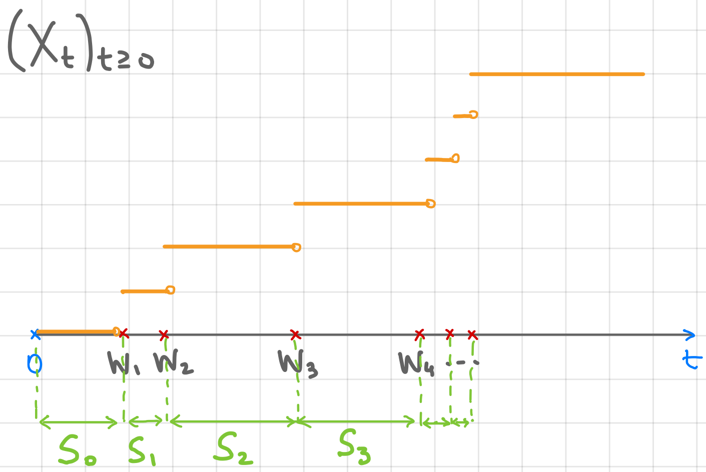
Birth processes and related differential equations

$P_n(t)$ satisfies the following system of differential eqs. with initial conditions

$$(*)\quad \begin{cases} P_0'(t) = -\lambda_0 P_0(t) & P_0(0) = 1 \\ P_1'(t) = -\lambda_1 P_1(t) + \lambda_0 P_0(t) & P_1(0) = 0 \\ P_2'(t) = -\lambda_2 P_2(t) + \lambda_1 P_1(t) & P_2(0) = 0 \\ \vdots & \vdots \\ P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t) & P_n(0) = 0 \\ \vdots & \vdots \end{cases}$$

Solving this system gives the p.m.f. of X_t for any t

Description of the birth processes via sojourn times



W_i – i-th "birth time" $\qquad$ S_i – "time between (i-1)-th birth and i-th birth"

$W_i = \sum_{l=0}^{i-1} S_l$ $\qquad$ ↳ sojourn times

Alternative way of characterizing $(X_t)_{t\geq 0}$:

- describe the distribution of $(S_i)_{i\geq 0}$
- desribe the jumps $X_{W_{i+1}} - X_{W_i}$

Description of the birth processes via sojourn times

Theorem

Let $(\lambda_k)_{k\geq 0}$ be a sequence of positive numbers. Let $(X_t)_{t\geq 0}$ be a non-decreasing right-continuous process, $X_0=0$, taking values in $\{0,1,2\ldots\}$. Let $(S_i)_{i\geq 0}$ be the sojourn times associated with $(X_t)_{t\geq 0}$, and define $W_\ell = \sum_{i=0}^{\ell-1} S_i$.

Then conditions

(a) $S_0, S_1, S_2, \ldots$ are independent exponential r.v.s of rate $\lambda_0, \lambda_1, \ldots$, $S_k \sim Exp(\lambda_k)$

(b) $X_{W_i} = i$

are equivalent to

(c) $(X_t)_{t\geq 0}$ is a pure birth process with parameters (λ_k)