MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

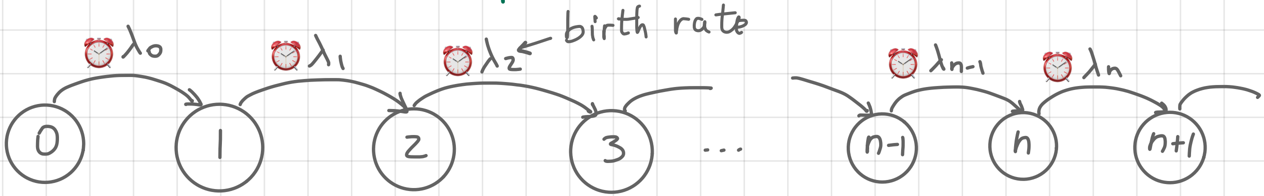
Today: Birth and death processes.
Strong Markov property.
Hitting probabilities

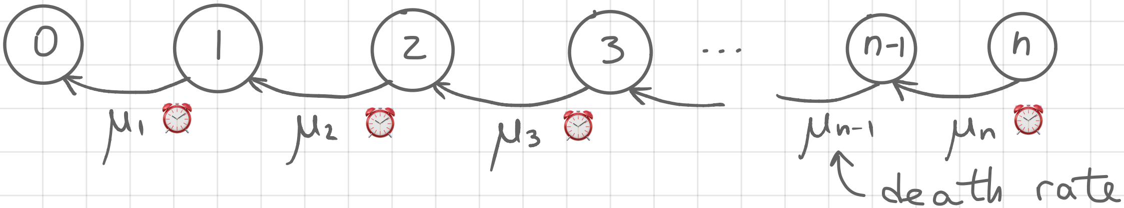
Next: PK 6.5, 6.6, Durrett 4.1

Week 2:

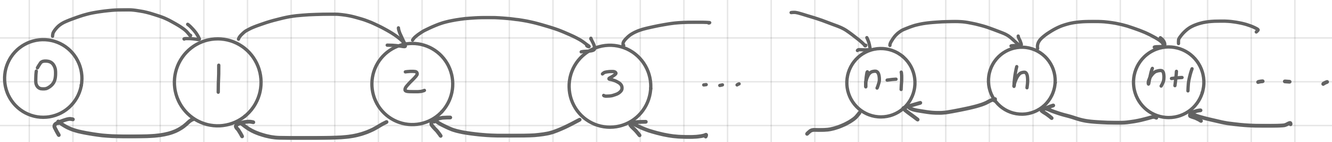
- HW1 due Friday, April 14 on Gradescope
- Important: Midterm 1 will take place on Friday, April 28

Birth and death processes





Combine both



Birth and death processes

Infinitesimal definition

Def. Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,2,\dots\}$ with stationary transition probabilities. Then $(X_t)_{t\geq 0}$ is called a birth and death process with birth rates (λ_k) and death rates (μ_k) if

1. $P_{i,i+1}(h) = \lambda_i h + o(h)$
2. $P_{i,i-1}(h) = \mu_i h + o(h)$
3. $P_{i,i}(h) = 1 - (\lambda_i + \mu_i) h + o(h)$
4. $P_{ij}(0) = \delta_{ij}$ $\left(P(X_0 = j \mid X_0 = i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \right)$
5. $\mu_0 = 0$, $\lambda_0 > 0$, $\lambda_i, \mu_i > 0$

Example: Linear growth with immigration

Dynamics of a certain population is described by the following principles:

during any small period of time of length h

- each individual gives birth to one new member with probability $\beta h + o(h)$ independently of other members;
- each individual dies with probability $\alpha h + o(h)$ independently of other members;
- one external member joins the population with probability $a h + o(h)$

Can be modeled as a Markov process

Example: Linear growth with immigration

Let $(X_t)_{t\geq 0}$ denote the size of the population at time t.

Using a similar argument as for the Yule/pure death models:

- $P_{n,n+1}(h) = \overbrace{n\beta h}^{\text{pure birth growth}} + ah + o(h)$ ← immigration growth

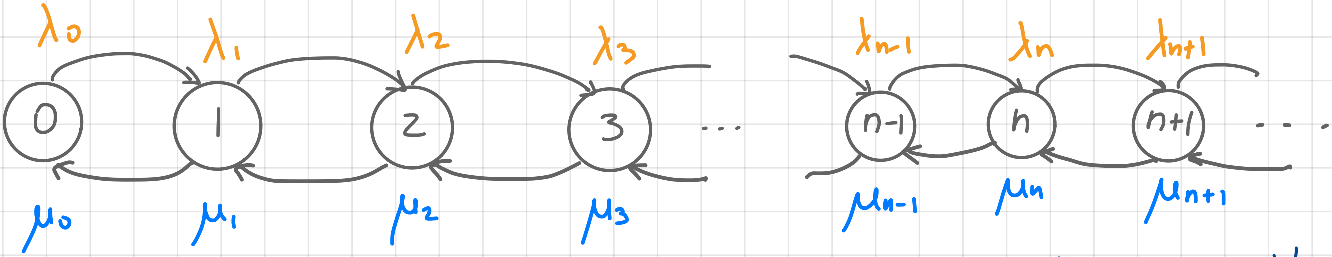
- $P_{n,n-1}(h) = n\alpha h + o(h)$
- $P_{n,n}(h) = 1 - (n\beta + a + n\alpha)h + o(h)$

↳ birth and death process with

$$\lambda_n = n\beta + a$$

$$\mu_n = n\alpha$$

Alternative (jump and hold) characterization



$\lambda = \mu = 1 \qquad \lambda' = \mu' = 2$

Sojourn times S_k are independent, $\quad \dfrac{\lambda}{\lambda+\mu} = \dfrac{\lambda'}{\lambda'+\mu'} = \dfrac{1}{2}$

$\mathrm{Exp}(2) \qquad \mathrm{Exp}(4)$

Each transition has two parts

- wait in state i for time $\sim \mathrm{Exp}(\lambda_i + \mu_i)$
- then choose where to go:

go $\to$ $(i+1)$ with probability $\dfrac{\lambda_i}{\lambda_i + \mu_i}$

go $(i-1) \leftarrow$ with pobability $\dfrac{\mu_i}{\lambda_i + \mu_i}$

Stopping times

Def (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a **stopping time** if the event

$$\{T \leq t\}$$

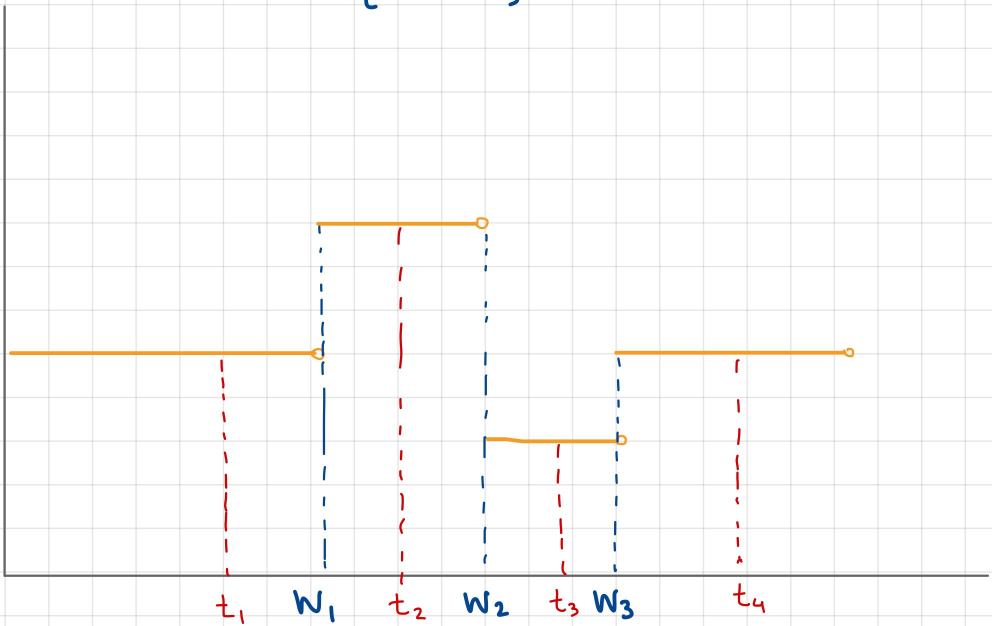
can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = i\}$ is a stopping time
2. W_k is a stopping time
3. $\sup\{t \geq 0 : X_t = i\}$ is not a stopping time

Stopping times

$\{T \leq t\}$



Strong Markov property

Theorem (no proof)

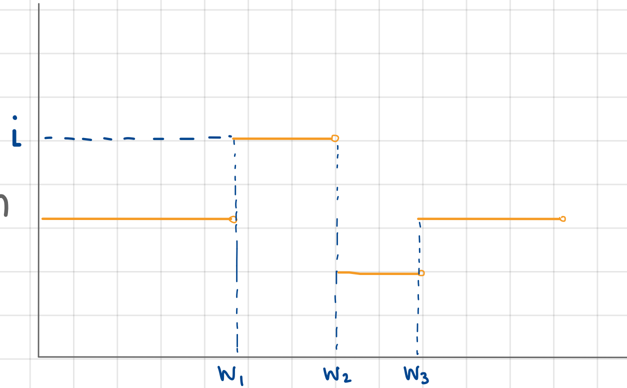
Let $(X_t)_{t\geq 0}$ be a MC, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T = i$,

$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from i.

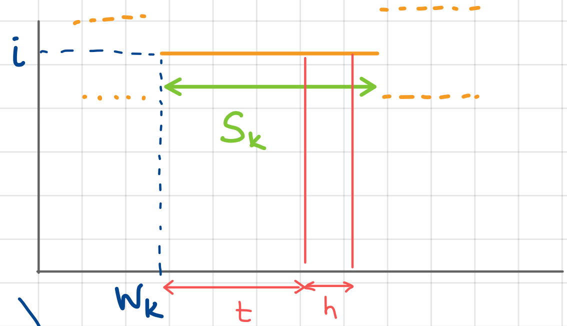
Example



$(X_{W_1+t})_{t\geq 0}$ has the same distribution as $(X_t)_{t\geq 0}$ conditioned on $X_0 = i$ and is indep. of what happened before

Alternative (jump and hold) characterization

"Proof"



↑ stopping time

Denote $G_i(t) := P(S_k > t \mid X_{W_k} = i)$

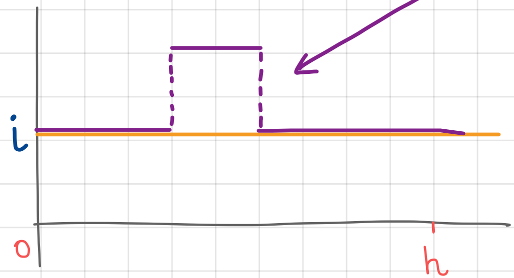
$$G_i(t+h) = P(S_k > t+h \mid X_{W_k} = i)$$

$$\overset{\text{SMarkov}}{=} P(\text{no jumps on } [0,t+h] \mid X_0 = i)$$

$$\overset{\text{Markov}}{=} P(\text{no jumps on } [0,t] \mid X_0 = i)\, P(\text{no jumps on } [0,h] \mid X_0 = i)$$

$$= P(S_0 > t \mid X_0 = i)\, P(S_0 > h \mid X_0 = i) = G_i(t)\Big(\overbrace{1-(\lambda_i+\mu_i)h+o(h)}^{P_{ii}(h)}\Big)$$

$$= G_i(t) - G_i(t)(\lambda_i+\mu_i)h + o(h)$$



↳ $G_i'(t) = -(\lambda_i+\mu_i)G_i(t)$, $G_i(0) = 1$

Alternative (jump and hold) characterization

"Proof" cont.

$$G_i'(t) = -(\lambda_i+\mu_i)\,G_i(t), \quad G_i(0)=1$$

$\hookrightarrow$ $G_i(t) = e^{-(\lambda_i+\mu_i)t} = P(S_k > t \mid X_{W_k}=i)$

✓ $\hookrightarrow$ $S_k \sim Exp(\lambda_i+\mu_i)$ (given that the process sojourns in i)

Suppose the process waits $Exp(\lambda_i+\mu_i)$, then
jumps to $i+1$ with probability $\lambda_i/(\lambda_i+\mu_i)$
to $i-1$ with probability $\mu_i/(\lambda_i+\mu_i)$

$$P_{i,i+1}(h) \text{ "=" } P(S_k \le h \mid X_{W_k}=i)\,P(\text{jump to } i+1)$$ ✓

$$= \left(1-e^{-(\lambda_i+\mu_i)h}\right)\frac{\lambda_i}{\lambda_i+\mu_i} = \left((\lambda_i+\mu_i)h+o(h)\right)\frac{\lambda_i}{\lambda_i+\mu_i} = \lambda_i h + o(h)$$

$$P_{i,i-1}(h) \text{ "=" } P(S_k \le h \mid X_{W_k}=i)\,P(\text{jump to } i-1) = \left((\lambda_i+\mu_i)h+o(h)\right)\frac{\mu_i}{\lambda_i+\mu_i} = \mu_i h + o(h)$$