MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

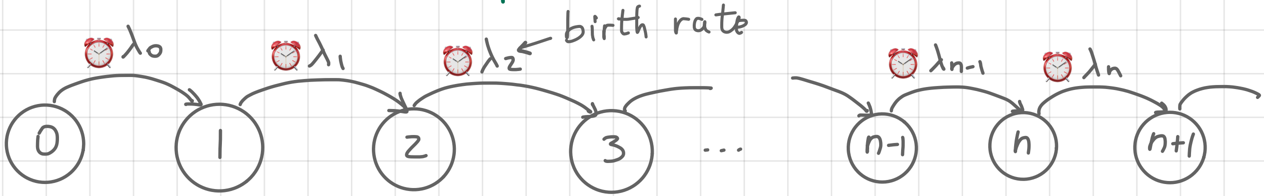
Today: Birth and death processes. Strong Markov property. Hitting probabilities

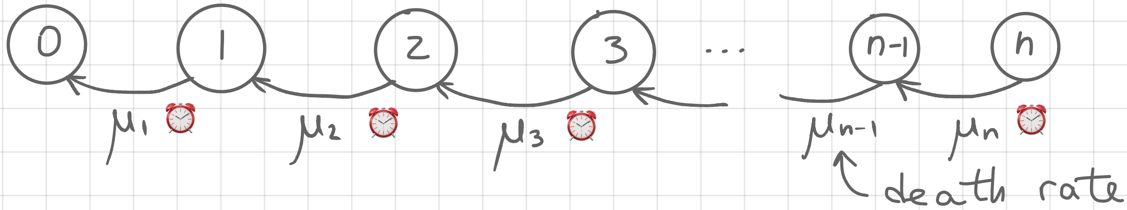
Next: PK 6.5, 6.6, Durrett 4.1

Week 2:

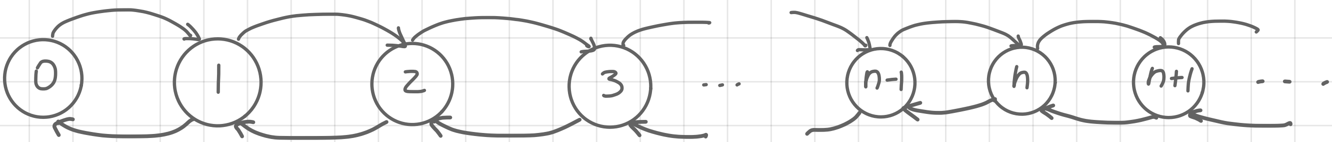
- HW1 due Friday, April 14 on Gradescope

Birth and death processes





Combine both



Birth and death processes

Infinitesimal definition

Def. Let $(X_t)_{t\geq 0}$ be a continuous time MC, $X_t \in \{0,1,2,\dots\}$ with stationary transition probabilities. Then $(X_t)_{t\geq 0}$ is called a birth and death process with birth rates (λ_k) and death rates (μ_k) if

1. $P_{i,i+1}(h) =$
2. $P_{i,i-1}(h) =$
3. $P_{i,i}(h) =$
4. $P_{ij}(0) =$ $\quad \left(P(X_0=j \mid X_0=i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \right)$
5. $\mu_0 = 0$, $\lambda_0 > 0$, $\lambda_i, \mu_i > 0$

Example: Linear growth with immigration

Dynamics of a certain population is described by the following principles:

during any small period of time of length h

- each individual gives birth to one new member with probability independently of other members;
- each individual dies with probability independently of other members;
- one external member joins the population with probability

Can be modeled as a Markov process

Example: Linear growth with immigration

Let $(X_t)_{t\geq 0}$ denote the size of the population.

Using a similar argument as for the Yule/pure death models:

- $P_{n,n+1}(h) =$

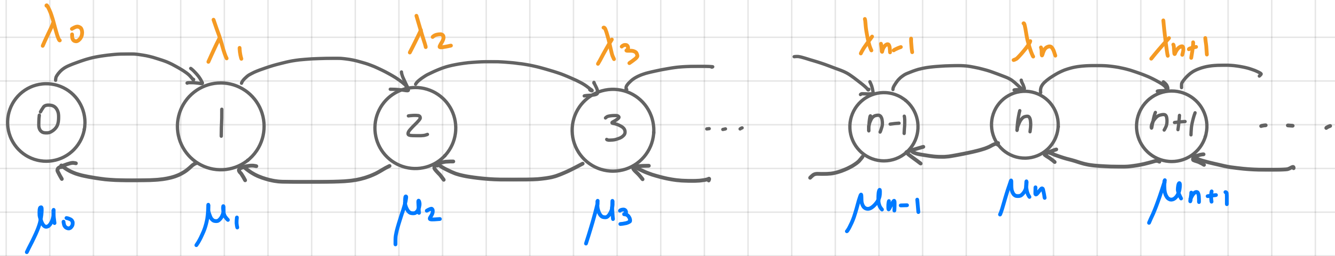
- $P_{n,n-1}(h) =$
- $P_{n,n}(h) =$

↳ birth and death process with

$\lambda_n =$

$\mu_n =$

Alternative (jump and hold) characterization



Sojourn times S_k are independent,

Each transition has two parts

- wait in state i for time ~
- then choose where to go:

go $\rightarrow$ $i+1$ with probability $\frac{\quad}{+}$

go $i-1$ $\leftarrow$ with pobability $\frac{\quad}{+}$

Stopping times

Def (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a stopping time if the event

$$\{T \leq t\}$$

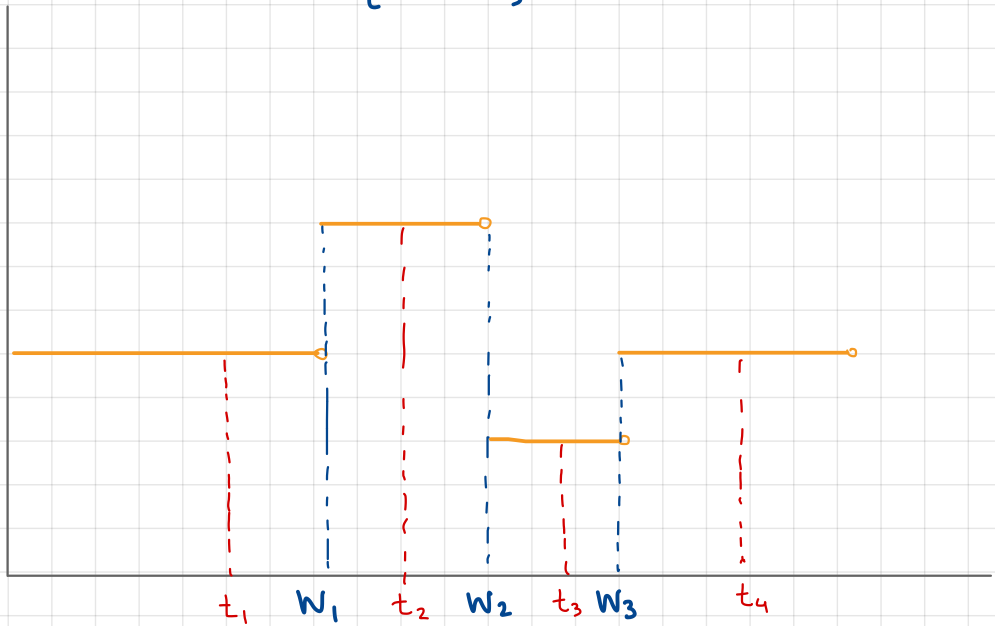
can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = i\}$ is a stopping time
2. W_k is a stopping time
3. $\sup\{t \geq 0 : X_t = i\}$ is not a stopping time

Stopping times

$\{T \leq t\}$



Strong Markov property

Theorem (no proof)

Let $(X_t)_{t\geq 0}$ be a MC, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T = i$,

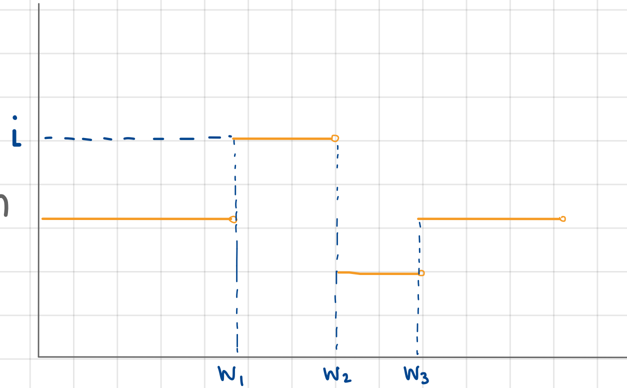
$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from i.

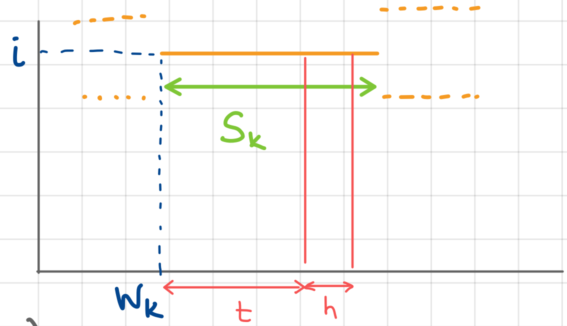
Example

$(X_{W_1+t})_{t\geq 0}$ has the same distribution as $(X_t)_{t\geq 0}$ conditioned on $X_0 = i$ and is indep. of what happened before



Alternative (jump and hold) characterization

"Proof"



Denote $G_i(t) := P(S_k > t \mid X_{W_k} = i)$

$G_i(t+h) = P(S_k > t+h \mid X_{W_k} = i)$

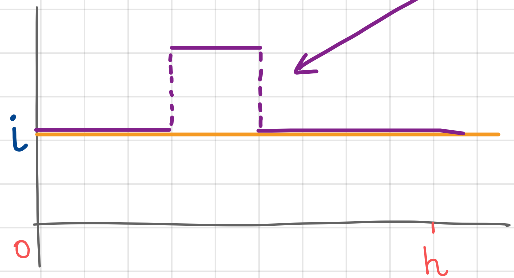
$\overset{\text{S Markov}}{=} P(\text{no jumps on } [0,t+h] \mid X_0 = i)$

$\overset{\text{Markov}}{=} P(\text{no jumps on } [0,t] \mid X_0 = i)\, P(\text{no jumps on } [0,h] \mid X_0 = i)$

$= P(S_0 > t \mid X_0 = i)\, P(S_0 > h \mid X_0 = i) = G_i(t)\big(1 - (\lambda_i + \mu_i)h + o(h)\big)$

$= G_i(t) - (\lambda_i + \mu_i) G_i(t) h + G_i(t) o(h)$

↳ $G_i'(t) = -(\lambda_i + \mu_i) G_i(t), \quad G_i(0) = 1$



Alternative (jump and hold) characterization

"Proof" cont.

$G_i'(t) = -(\lambda_i+\mu_i)G_i(t)$, $G_i(0)=1$

$\hookrightarrow G_i(t) = e^{-(\lambda_i+\mu_i)t} = P(S_k>t \mid X_{W_k}=i)$

✓ $\hookrightarrow S_k \sim \mathrm{Exp}(\lambda_i+\mu_i)$ (given that the process sojourns in i)

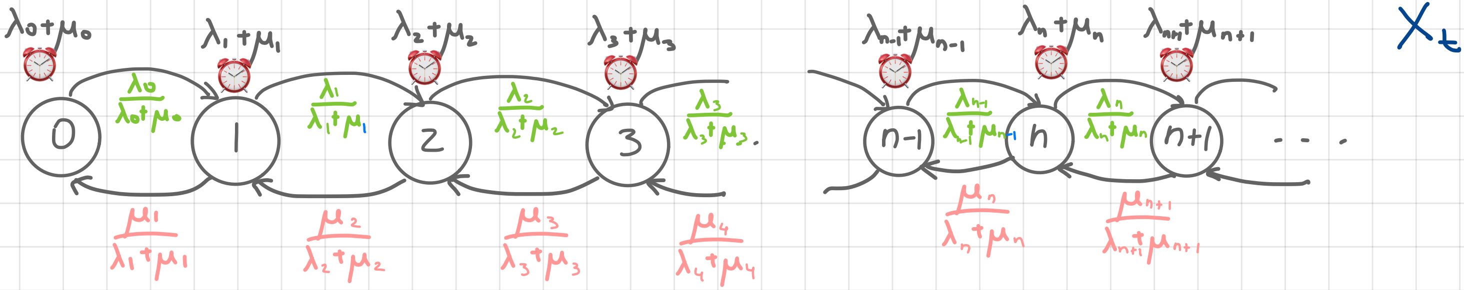
Suppose the process waits $\mathrm{Exp}(\lambda_i+\mu_i)$, then
jumps to $i+1$ with probability $\lambda_i/(\lambda_i+\mu_i)$
to $i-1$ with probability $\mu_i/(\lambda_i+\mu_i)$

$$P_{i,i+1}(h) \text{"="} P(S_k \le h \mid X_{W_k}=i)\,P(\text{jump to } i+1)$$ ✓

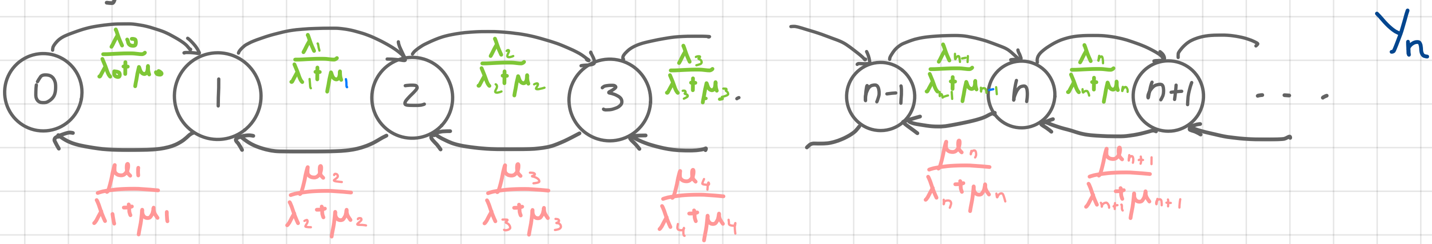
$$= \left(1-e^{-(\lambda_i+\mu_i)h}\right)\frac{\lambda_i}{\lambda_i+\mu_i} = \left((\lambda_i+\mu_i)h+o(h)\right)\frac{\lambda_i}{\lambda_i+\mu_i} = \lambda_i h+o(h)$$

$$P_{i,i-1}(h) \text{"="} P(S_k \le h \mid X_{W_k}=i)\,P(\text{jump to } i-1) = \left((\lambda_i+\mu_i)h+o(h)\right)\frac{\mu_i}{\lambda_i+\mu_i} = \mu_i h+o(h)$$

Related discrete time MC.



Def. Let $(X_t)_{t\geq 0}$ be a continuous time MC, let $W_n, n\geq 0$, be the corresponding waiting (arrival, jump) times. Then we call $(Y_n)_{n\geq 0}$ defined by the jump chain of $(X_t)_{t\geq 0}$.



random walk

Absorption probabilities for B&D processes

Let $(X_t)_{t\geq 0}$ be a birth and death process, and assume that the state 0 is absorbing, $\lambda_0 = 0$. Then

$P((X_t)_{t\geq 0} \text{ gets absorbed in } 0 \mid X_0 = i)$

↳ use the first step analysis to compute the absorption probabilities for $(Y_n)_{n\geq 0}$ (and for $(X_t)_{t\geq 0}$)

Denote $u_i = P(Y_n \text{ is absorded in } 0 \mid Y_0 = i)$

Then

Absorption probabilities for B&D processes

$$u_0 = 1, \qquad u_n = \frac{\mu_n}{\lambda_n + \mu_n} u_{n-1} + \frac{\lambda_n}{\lambda_n + \mu_n} u_{n+1}$$

Rewrite

$$(\lambda_n + \mu_n) u_n = \mu_n u_{n-1} + \lambda_n u_{n+1}$$

$$\lambda_n (u_{n+1} - u_n) = \mu_n (u_n - u_{n-1})$$

$$u_{n+1} - u_n = \frac{\mu_n}{\lambda_n} (u_n - u_{n-1})$$

$$= \underbrace{\frac{\mu_n}{\lambda_n} \cdot \frac{\mu_{n-1}}{\lambda_{n-1}} \cdot \dots \cdot \frac{\mu_1}{\lambda_1}}_{\rho_n} (u_1 - \overset{=1}{u_0})$$

$$(*) \quad u_{n+1} - u_n = \rho_n (u_1 - 1)$$

Note that $\sum_{k=1}^{n-1} (u_{k+1} - u_k) = u_n - u_1 = (u_1 - 1) \sum_{n=1}^{n-1} \rho_n \qquad (**)$

If $\sum_{n=1}^{\infty} \rho_n = \infty$, then $u_1 = 1$ and from $(*)$ $u_n = 1 \ \forall n \geq 0$.

Absorption probabilities for B&D processes

Let $\sum_{k=1}^{\infty} \rho_k < \infty$. We are looking for the minimal solution that satisfies $u_n \in [0,1]$ $\forall n$. We rewrite (**) as

Choose smallest $u_1 \in [0,1]$ for which