MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

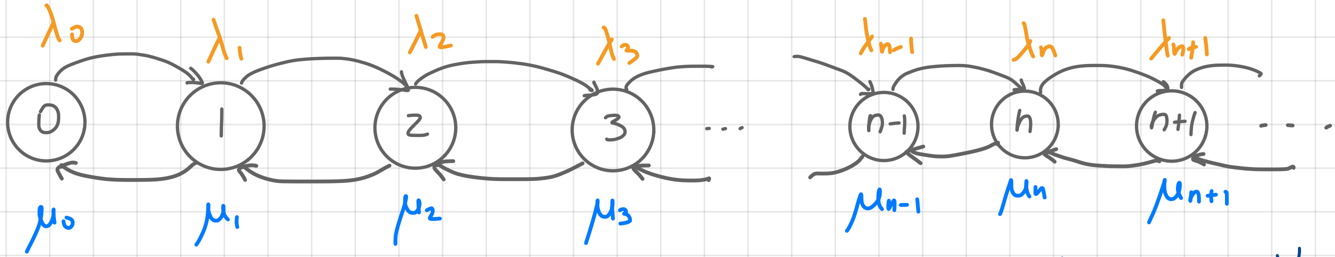
Today: Hitting probabilities.
Absorption times. General CTMC.
Matrix exponentials

Next: PK 6.5, 6.6, Durrett 4.1

Week 2:

- HW1 due Friday, April 14 on Gradescope
- Important: Midterm 1 will take place on Friday, April 28

Alternative (jump and hold) characterization



$\lambda = \mu = 1 \qquad \lambda' = \mu' = 2$

Sojourn times S_k are independent, $\quad \dfrac{\lambda}{\lambda+\mu} = \dfrac{\lambda'}{\lambda'+\mu'} = \dfrac{1}{2}$

$\text{Exp}(2) \qquad \text{Exp}(4)$

Each transition has two parts

- wait in state i for time $\sim \text{Exp}(\lambda_i + \mu_i)$
- then choose where to go:

go $\to$ $(i+1)$ with probability $\dfrac{\lambda_i}{\lambda_i + \mu_i}$

go $(i-1) \leftarrow$ with pobability $\dfrac{\mu_i}{\lambda_i + \mu_i}$

Stopping times

Def (Informal). Let $(X_t)_{t\geq 0}$ be a stochastic process and let $T \geq 0$ be a random variable. We call T a **stopping time** if the event

$$\{T \leq t\}$$

can be determined from the knowledge of the process up to time t (i.e., from $\{X_s : 0 \leq s \leq t\}$)

Examples: Let $(X_t)_{t\geq 0}$ be right-continuous

1. $\min\{t \geq 0 : X_t = i\}$ is a stopping time
2. W_k is a stopping time
3. $\sup\{t \geq 0 : X_t = i\}$ is not a stopping time

Strong Markov property

Theorem (no proof)

Let $(X_t)_{t\geq 0}$ be a MC, let T be a stopping time of $(X_t)_{t\geq 0}$. Then, conditional on $T<\infty$ and $X_T = i$,

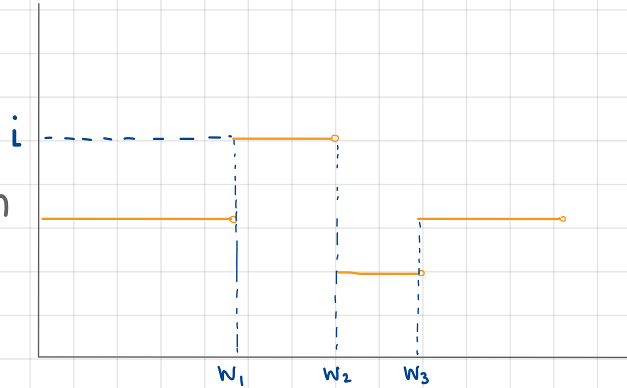
$$(X_{T+t})_{t\geq 0}$$

(i) is independent of $\{X_s,\ 0\leq s\leq T\}$

(ii) has the same distribution as $(X_t)_{t\geq 0}$ starting from i.

Example

$(X_{W_1+t})_{t\geq 0}$ has the same distribution as $(X_t)_{t\geq 0}$ conditioned on $X_0 = i$ and is indep. of what happened before



Alternative (jump and hold) characterization

"Proof" cont.

$$G_i'(t) = -(\lambda_i+\mu_i)G_i(t), \quad G_i(0)=1$$

$\hookrightarrow G_i(t) = e^{-(\lambda_i+\mu_i)t} = P(S_k>t \mid X_{W_k}=i)$

✓ $\hookrightarrow S_k \sim \mathrm{Exp}(\lambda_i+\mu_i)$ (given that the process sojourns in i)

Suppose the process waits $\mathrm{Exp}(\lambda_i+\mu_i)$, then

jumps to $i+1$ with probability $\lambda_i/(\lambda_i+\mu_i)$

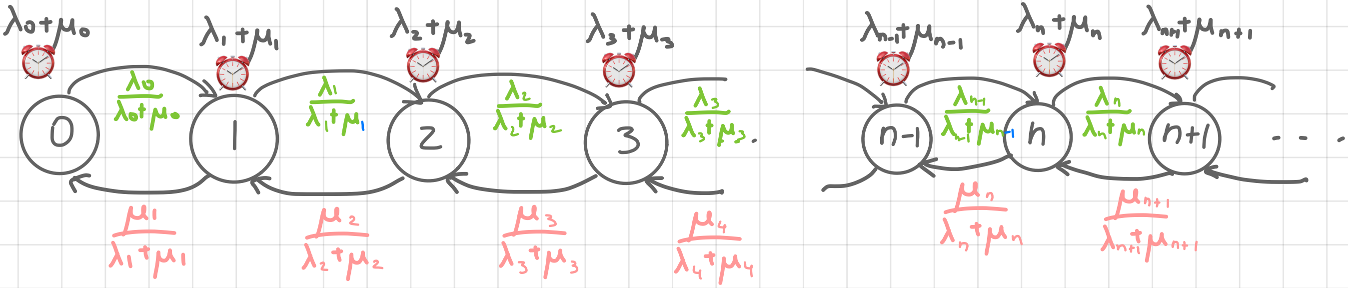
to $i-1$ with probability $\mu_i/(\lambda_i+\mu_i)$

$$P_{i,i+1}(h) \text{"="} P(S_k \le h \mid X_{W_k}=i)\,P(\text{jump to } i+1)$$ ✓

$$= \left(1-e^{-(\lambda_i+\mu_i)h}\right)\frac{\lambda_i}{\lambda_i+\mu_i} = \left((\lambda_i+\mu_i)h+o(h)\right)\frac{\lambda_i}{\lambda_i+\mu_i} = \lambda_i h + o(h)$$

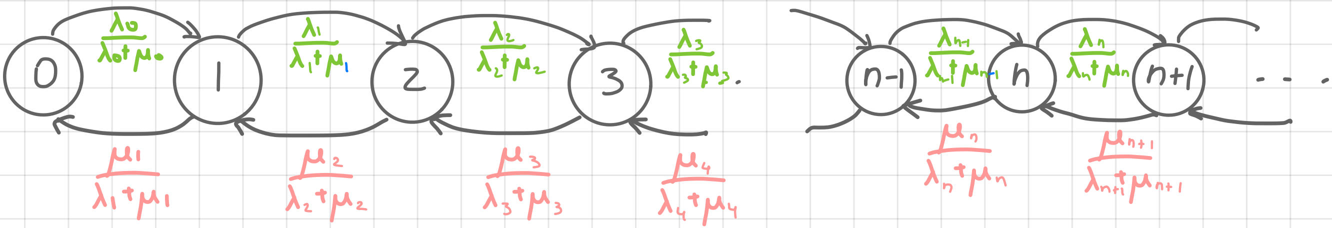
$$P_{i,i-1}(h) \text{"="} P(S_k \le h \mid X_{W_k}=i)\,P(\text{jump to } i-1) = \left((\lambda_i+\mu_i)h+o(h)\right)\frac{\mu_i}{\lambda_i+\mu_i} = \mu_i h + o(h)$$

Related discrete time MC.



X_t

Def. Let $(X_t)_{t\geq 0}$ be a continuous time MC, let $W_n, n\geq 0$, be the corresponding waiting (arrival, jump) times. Then we call $(Y_n)_{n\geq 0}$ defined by the jump chain of $(X_t)_{t\geq 0}$.



Y_n

↑ random walk

Absorption probabilities for B&D processes

Let $(X_t)_{t\geq 0}$ be a birth and death process, and assume that the state 0 is absorbing, $\lambda_0 = 0$. Then

$P((X_t)_{t\geq 0}$ gets absorbed in $0 \mid X_0 = i)$

↳ use the first step analysis to compute the absorption probabilities for $(Y_n)_{n\geq 0}$ (and for $(X_t)_{t\geq 0}$)

Denote $u_i = P(Y_n$ is absorded in $0 \mid Y_0 = i)$

Then

Absorption probabilities for B&D processes

$u_0 = 1$, $\quad u_n = \dfrac{\mu_n}{\lambda_n + \mu_n} u_{n-1} + \dfrac{\lambda_n}{\lambda_n + \mu_n} u_{n+1}$

Rewrite $\quad (\lambda_n + \mu_n) u_n = \mu_n u_{n-1} + \lambda_n u_{n+1}$

$$\lambda_n (u_{n+1} - u_n) = \mu_n (u_n - u_{n-1})$$

$$u_{n+1} - u_n = \frac{\mu_n}{\lambda_n} (u_n - u_{n-1})$$

$$= \underbrace{\frac{\mu_n}{\lambda_n} \cdot \frac{\mu_{n-1}}{\lambda_{n-1}} \cdot \cdots \cdot \frac{\mu_1}{\lambda_1}}_{\rho_n} (u_1 - \overset{= 1}{u_0})$$

$(*) \quad u_{n+1} - u_n = \rho_n (u_1 - 1)$

Note that $\displaystyle\sum_{k=1}^{n-1} (u_{k+1} - u_k) = u_n - u_1 = (u_1 - 1) \sum_{n=1}^{n-1} \rho_n \qquad (**)$

If $\displaystyle\sum_{n=1}^{\infty} \rho_n = \infty$, then $u_1 = 1$ and from $(*)$ $u_n = 1 \ \forall \ n \geq 0$.

Absorption probabilities for B&D processes

Let $\sum_{k=1}^{\infty} \rho_k < \infty$. We are looking for the minimal solution that satisfies $u_n \in [0,1]$ $\forall n$. We rewrite (**) as

Choose smallest $u_1 \in [0,1]$ for which

Mean time until absorption

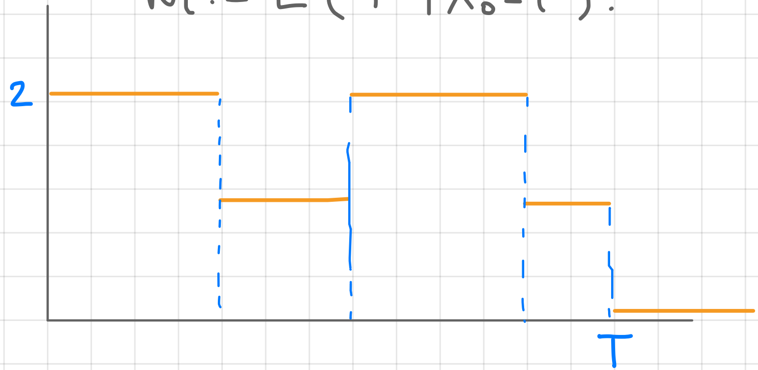
Let $(X_t)_{t\geq 0}$ be a birth and death process. Denote

$T = \min\{t \geq 0 : X_t = 0\}$ absorption time and

$w_i := E(T \mid X_0 = i)$.

Let $(Y_n)_{n\geq 0}$ be the jump chain for $(X_t)_{t\geq 0}$.

$N := \min\{n \geq 0 : Y_n = 0\}$



Then

$$w_i = E\left(\sum_{k=0}^{N-1} S_k \mid X_0 = i\right) = \frac{1}{\lambda_i + \mu_i} + E\left(\sum_{k=1}^{N-1} S_k \mid X_0 = i\right)$$

$$= \frac{1}{\lambda_i + \mu_i} + E\left(\sum_{k=1}^{N-1} S_k \mid X_0 = i, Y_1 = i+1\right) P(Y_1 = i+1 \mid Y_0 = i)$$

$$+ E\left(\sum_{k=1}^{N-1} S_k \mid X_0 = i, Y_1 = i-1\right) P(Y_1 = i-1 \mid Y_0 = i)$$

Mean time until absorption

$$\begin{cases} w_i = \dfrac{1}{\lambda_i + \mu_i} + \dfrac{\lambda_i}{\lambda_i + \mu_i} w_{i+1} + \dfrac{\mu_i}{\lambda_i + \mu_i} w_{i-1}, \\ w_0 = 0 \end{cases}$$

$$w_i = \begin{cases} \displaystyle\sum_{j=1}^{\infty} \frac{1}{\lambda_j \rho_j} + \sum_{k=1}^{i-1} \rho_k \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j \rho_j}, \text{ if } \sum_{j=1}^{\infty} \frac{1}{\lambda_j \rho_j} < \infty \\ \infty, \text{ if } \displaystyle\sum_{j=1}^{\infty} \frac{1}{\lambda_j \rho_j} = \infty \end{cases}$$

First step analysis for birth and death processes

Summary:

Let $(X_t)_{t\geq 0}$ be a birth and death process of rates $((\lambda_i,\mu_i))_{i\geq 0}$ with $\lambda_0=0$ (state 0 absorbing).

Denote $T=\min\{t: X_t=0\}$, $u_i = P(X_t \text{ gets absorbed in } 0 \mid X_0=i)$

$W_i = E(T \mid X_0=i)$ and $\rho_j = \dfrac{\mu_1\mu_2\cdots\mu_j}{\lambda_1\lambda_2\cdots\lambda_j}$. Then

$$u_i = \begin{cases} \dfrac{\sum_{j=i}^{\infty}\rho_j}{1+\sum_{j=1}^{\infty}\rho_j}, & \text{if } \sum_{j=1}^{\infty}\rho_j<\infty \\ 1, & \text{if } \sum_{j=1}^{\infty}\rho_j=\infty \end{cases}$$

$$W_i = \begin{cases} \sum_{j=1}^{\infty}\dfrac{1}{\lambda_j\rho_j}+\sum_{k=1}^{i-1}\rho_k\sum_{j=k+1}^{\infty}\dfrac{1}{\lambda_j\rho_j}, & \text{if } \sum_{j=1}^{\infty}\dfrac{1}{\lambda_j\rho_j}<\infty \\ \infty, & \text{if } \sum_{j=1}^{\infty}\dfrac{1}{\lambda_j\rho_j}=\infty \end{cases}$$

Birth and death processes. Results

- infinitesimal transition probability description
- sojourn time description (jump and hold)

  sojourn times are independent exponential r.v.s

  $P(i \to i+1) = \frac{\lambda_i}{\lambda_i + \mu_i}$, $P(i \to i-1) = \frac{\mu_i}{\lambda_i + \mu_i}$

- system of differential equations for pure birth/death

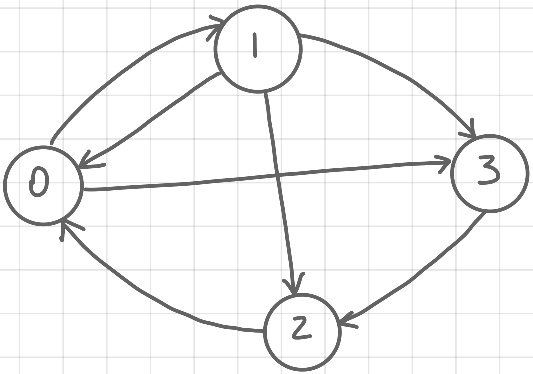
  e.g. $P_i'(t) = -\lambda_i P_i(t) + \lambda_{i-1} P_{i-1}(t)$

- distributions of X_t for linear birth (geometric) and linear death (binomial) processes
- first step analysis giving absorption probabilities and mean time to absorption
- explosion, Strong Markov property etc.

General continuous time MC

Assume for simplicity that the state space is finite

birth and death process



general MC

How to define? How to analyze?

Q-matrices (infinitesimal generators)

Let $S = \{0, 1, \dots, N\}$. We call $Q = (q_{ij})_{i,j=0}^{N}$ a Q-matrix if Q satisfies the following conditions:

(a)

(b)

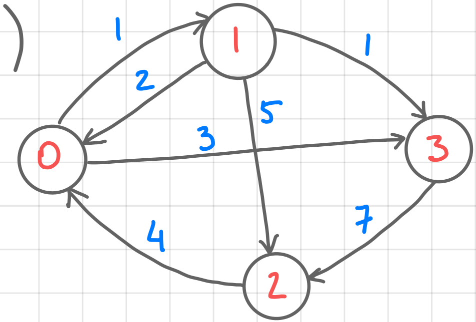
(c)

Examples

(a)

$$Q = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

(b)



$$\begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{pmatrix} \end{array}$$

Matrix exponentials

Let $Q = (q_{ij})_{i,j=1}^{N}$ be a matrix. Then the series converges componentwise, and we denote

its sum $\sum_{k=0}^{\infty} \frac{Q^k}{k!} =:$, the matrix exponential of Q.

In particular, we can define for $t \geq 0$.

Thm. Define $P(t) = e^{tQ}$. Then

(i) for all s, t

(ii) $(P(t))_{t\geq 0}$ is the unique solution to the equations

$$\begin{cases} \frac{d}{dt} P(t) = \\ P(0) = \end{cases}$$

, and

$$\begin{cases} \frac{d}{dt} P(t) = \\ P(0) = \end{cases}$$

Matrix exponentials

Properties are easy to remember $\longrightarrow$ scalar exponential

(i) $e^{(t+s)Q} = e^{tQ} e^{sQ} = e^{sQ} e^{tQ}$ $\quad \left(e^{(t+s)\alpha} = e^{t\alpha} e^{s\alpha} \right)$

(note that in general $AB \neq BA$ for matrices A, B)

(ii) $\frac{d}{dt} e^{tQ} = Q e^{tQ} = e^{tQ} Q$ $\quad \left(\frac{d}{dt} e^{t\alpha} = \alpha e^{t\alpha} \right)$

$e^{0 \cdot Q} = I$ $\quad (e^0 = 1)$

Example

(a) $Q_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

(b) $Q_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Matrix exponentials

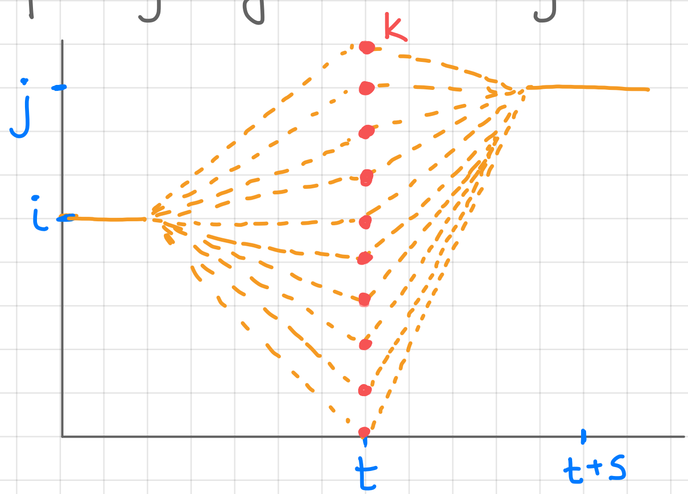
Results on the previous slide hold for any matrix Q.

<u>Thm</u>. Matrix Q is a Q-matrix

iff $P(t) = e^{tQ}$ is a stochastic matrix $\forall t$

<u>Remarks</u> The semigroup property gives entrywise

$$P_{ij}(t+s) = [P(t)P(s)]_{ij}$$



(if you think about MC →
Chapman-Kolmogorov)