MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: Absorption times. General CTMC. Matrix exponentials

Next: 6.6, Durrett 4.1

Week 3:

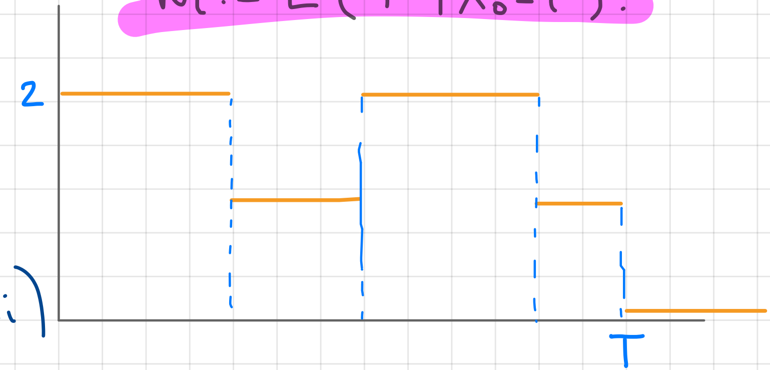
- HW2 due Friday, April 21 on Gradescope
- No in-person lecture on Friday, April 21

Mean time until absorption

Let $(X_t)_{t\geq 0}$ be a birth and death process. Denote

$T = \min\{t \geq 0 : X_t = 0\}$ absorption time and $w_i := E(T \mid X_0 = i)$.

Let $(Y_n)_{n\geq 0}$ be the jump chain for $(X_t)_{t\geq 0}$.



$$N := \min\{n \geq 0 : Y_n = 0\}$$

Then $$T = \sum_{k=0}^{N-1} S_k$$

$$w_i = E\left(\sum_{k=0}^{N-1} S_k \,\Big|\, X_0 = i\right) = \overbrace{\frac{1}{\lambda_i+\mu_i}}^{= E(S_0 \mid X_0 = i)} + E\left(\sum_{k=1}^{N-1} S_k \,\Big|\, X_0 = i\right)$$

$$= \frac{1}{\lambda_i+\mu_i} + \underbrace{E\left(\sum_{k=1}^{N-1} S_k \,\Big|\, \overset{Y_0=i}{X_0 = i},\, Y_1 = i+1\right)}_{\text{SMP} \; \parallel \; w_{i+1}} \overbrace{P(Y_1 = i+1 \mid Y_0 = i)}^{\frac{\lambda_i}{\lambda_i+\mu_i}}$$

$$+ \overbrace{E\left(\sum_{k=1}^{N-1} S_k \,\Big|\, X_0 = i,\, Y_1 = i-1\right)}^{\text{SMP} \; = \; w_{i-1}} \overbrace{P(Y_1 = i-1 \mid Y_0 = i)}^{\frac{\mu_i}{\lambda_i+\mu_i}}$$

Mean time until absorption

$$\begin{cases} w_i = \dfrac{1}{\lambda_i+\mu_i} + \dfrac{\lambda_i}{\lambda_i+\mu_i} w_{i+1} + \dfrac{\mu_i}{\lambda_i+\mu_i} w_{i-1}, \\ w_0 = 0 \end{cases}$$

$$w_i = \begin{cases} \displaystyle\sum_{j=1}^{\infty} \frac{1}{\lambda_j \rho_j} + \sum_{k=1}^{i-1} \rho_k \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j \rho_j}, & \text{if } \displaystyle\sum_{j=1}^{\infty} \frac{1}{\lambda_j \rho_j} < \infty \\ \infty, & \text{if } \displaystyle\sum_{j=1}^{\infty} \frac{1}{\lambda_j \rho_j} = \infty \end{cases}$$

First step analysis for birth and death processes

Summary:

Let $(X_t)_{t\geq 0}$ be a birth and death process of rates $((\lambda_i, \mu_i))_{i\geq 0}$ with $\lambda_0 = 0$ (state 0 absorbing).

Denote $T = \min\{t : X_t = 0\}$, $u_i = P(X_t \text{ gets absorbed in } 0 \mid X_0 = i)$

$w_i = E(T \mid X_0 = i)$ and $\rho_j = \dfrac{\mu_1 \mu_2 \cdots \mu_j}{\lambda_1 \lambda_2 \cdots \lambda_j}$. Then

$$u_i = \begin{cases} \dfrac{\sum_{j=i}^{\infty} \rho_j}{1 + \sum_{j=1}^{\infty} \rho_j}, & \text{if } \sum_{j=1}^{\infty} \rho_j < \infty \\ 1, & \text{if } \sum_{j=1}^{\infty} \rho_j = \infty \end{cases}$$

$$w_i = \begin{cases} \sum_{j=1}^{\infty} \dfrac{1}{\lambda_j \rho_j} + \sum_{k=1}^{i-1} \rho_k \sum_{j=k+1}^{\infty} \dfrac{1}{\lambda_j \rho_j}, & \text{if } \sum_{j=1}^{\infty} \dfrac{1}{\lambda_j \rho_j} < \infty \\ \infty, & \text{if } \sum_{j=1}^{\infty} \dfrac{1}{\lambda_j \rho_j} = \infty \end{cases}$$

Birth and death processes. Results

- infinitesimal transition probability description
- sojourn time description (jump and hold)

  sojourn times are independent exponential r.v.s

  $$P(i \to i+1) = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P(i \to i-1) = \frac{\mu_i}{\lambda_i + \mu_i}$$

- system of differential equations for pure birth/death

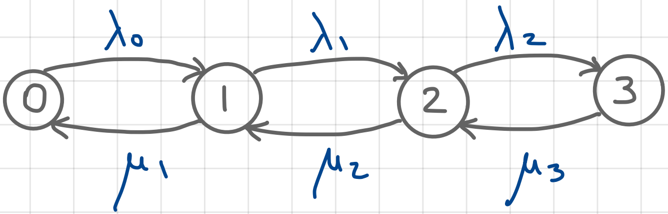
  e.g. $$P_i'(t) = -\lambda_i P_i(t) + \lambda_{i-1} P_{i-1}(t)$$

- distributions of X_t for linear birth (geometric) and linear death (binomial) processes
- first step analysis giving absorption probabilities and mean time to absorption
- explosion, Strong Markov property etc.

General continuous time MC

Assume for simplicity that the state space is finite

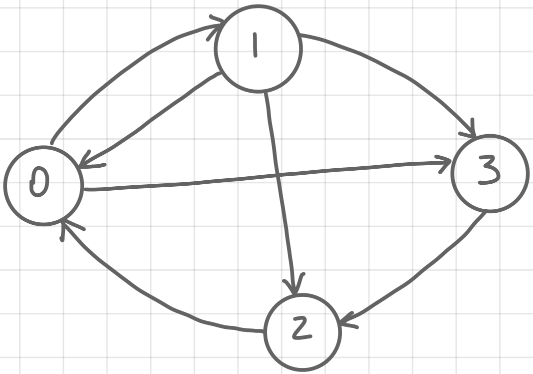
$(X_t)_{t \geq 0}$ is right-continuous



birth and death process

$P_{i,i+1}(h) = \lambda_i h + o(h)$

$P_{ii}(h) = 1 - \lambda_i h + o(h)$



general MC

$P_{ij}(s) := P(X_{t+s} = j \mid X_t = i) = P(X_s = j \mid X_0 = i)$

$i \neq j \qquad P_{ij}(h) = q_{ij} \cdot h + o(h) \quad \text{as } h \downarrow 0$

How to define? How to analyze?

Q-matrices (infinitesimal generators)

Let $S=\{0,1,\dots,N\}$. We call $Q=(q_{ij})_{i,j=0}^{N}$ a Q-matrix if Q satisfies the following conditions:

(a) $0 \le -q_{ii} < \infty$ for all i

(b) $q_{ij} \ge 0$ for all $i \ne j$

(c) $\sum_j q_{ij} = 0$ for all i

$$q_i := \sum_{j\ne i} q_{ij}$$

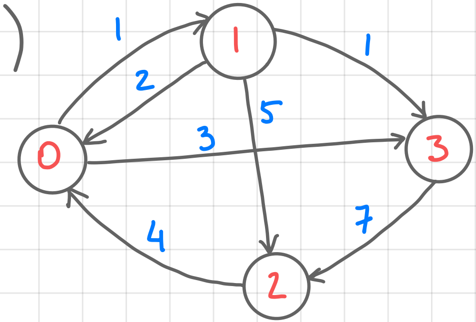
$$q_{ii} = -q_i$$

Examples

(a)

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -7 & 5 \\ 0 & 2 & -2 \end{pmatrix}$$

(b)



$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} -4 & 1 & 0 & 3 \\ 2 & -8 & 5 & 1 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 7 & -7 \end{pmatrix} \end{array}$$

Matrix exponentials

$Q^0 = I$

Let $Q = (q_{ij})_{i,j=1}^N$ be a matrix. Then the series $\sum_{k=0}^{\infty} \frac{Q^k}{k!}$ converges componentwise, and we denote

its sum $\sum_{k=0}^{\infty} \frac{Q^k}{k!} =: e^Q$, the matrix exponential of Q.

In particular, we can define $e^{tQ} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!}$ for $t \geq 0$.

Thm. Define $P(t) = e^{tQ}$. Then

semigroup

(i) $P(t+s) = P(t)P(s)$ for all $s, t > 0$

(ii) $(P(t))_{t \geq 0}$ is the unique solution to the equations

$$\begin{cases} \frac{d}{dt} P(t) = P(t)Q, \\ P(0) = I \end{cases} \quad \text{and} \quad \begin{cases} \frac{d}{dt} P(t) = QP(t) \\ P(0) = I \end{cases}$$

Matrix exponentials

Properties are easy to remember $\longrightarrow$ scalar exponential

(i) $e^{(t+s)Q} = e^{tQ} e^{sQ} = e^{sQ} e^{tQ}$ $\quad \left(e^{(t+s)\alpha} = e^{t\alpha} e^{s\alpha} \right)$

(note that in general $AB \neq BA$ for matrices A, B)

(ii) $\frac{d}{dt} e^{tQ} = Q e^{tQ} = e^{tQ} Q$ $\quad \left(\frac{d}{dt} e^{t\alpha} = \alpha e^{t\alpha} \right)$

$e^{0 \cdot Q} = I$ $\quad (e^0 = 1)$

Example

(a) $Q_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Q_1^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $e^{tQ_1} = I + t \cdot Q_1 + 0 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

(b) $Q_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $e^{tQ_2} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$

Matrix exponentials

Results on the previous slide hold for any matrix Q.

<u>Thm</u>. Matrix Q is a Q-matrix

iff $P(t) = e^{tQ}$ is a stochastic matrix $\forall t$

$$P_{ij}(t) \geq 0, \qquad \sum_j P_{ij}(t) = 1 \quad \forall i \text{ and } \forall t \geq 0$$

<u>Remarks</u> The semigroup property gives entrywise

$$P_{ij}(t+s) = [P(t)P(s)]_{ij}$$

$$\sum_{k=0}^{N} P_{ik}(t) P_{kj}(s)$$

(if you think about MC →

Chapman-Kolmogorov)