MATH180C: Introduction to Stochastic Processes II

https://mathweb.ucsd.edu/~ynemish/teaching/180c

Today: FSA for general MC.
Kolmogorov equations
Next: PK 6.3, 6.6, Durrett 4.2

Week 3:

- HW2 due Friday, April 21 on Gradescope
- No in-person lecture on Friday, April 21

Absorption probabilities for finite state chains

(X_t)

$$Q = \begin{array}{c} \\ 0 \\ \vdots \\ K-1 \\ K \\ \vdots \\ N \end{array}\begin{array}{c} \begin{array}{ccc|ccc} 0 & \cdots & K-1 & K & \cdots & N \end{array} \\ \left(\begin{array}{ccc|ccc} -q_0 & & & \cdots & q_{ij} & \\ \vdots & & & & \vdots & \\ q_{ij} & \cdots & -q_{K-1} & & & \\ \hline & & & 0 & & 0 \\ & 0 & & & \ddots & \\ & & & & & 0 \end{array}\right) \end{array}$$

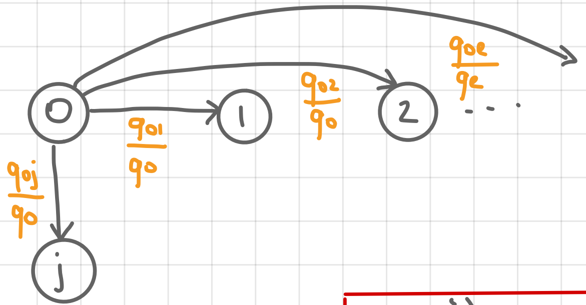
Let $i \in \{0, \ldots, K-1\}$, $j \in \{K, \ldots, N\}$.

Let $\mathbf{M} = \min\{n : Y_n \in \{K, \ldots, N\}\}$

Denote $u_i^{(j)} = P(Y_{\mathbf{M}} = j \mid X_0 = i)$.

Then FSA leads to the system

Jump chain



$$u_i^{(j)} = P(Y_{\mathbf{M}} = j \mid Y_0 = i)$$

$$= \sum_{\substack{\ell=0 \\ \ell \neq i}}^{N} P(Y_M = j \mid Y_0 = i, Y_1 = \ell)\, P(Y_1 = \ell \mid Y_0 = i)$$

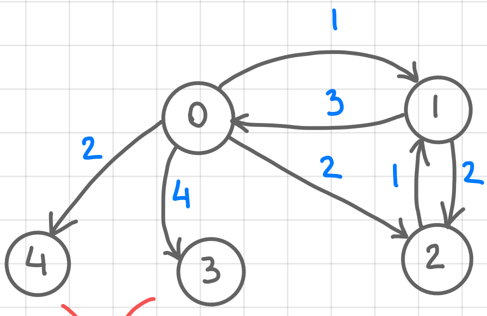
$$= \sum_{\substack{\ell=0 \\ \ell \neq i}}^{K-1} \underbrace{P(Y_M = j \mid Y_1 = \ell)}_{u_\ell^{(j)}} \frac{q_{i\ell}}{q_i} + \underbrace{P(Y_M = j \mid Y_1 = j)}_{1} \frac{q_{ij}}{q_i}$$

$$\boxed{u_i^{(j)} = \frac{q_{ij}}{q_i} + \sum_{\substack{\ell=0 \\ \ell \neq i}}^{K-1} \frac{q_{i\ell}}{q_i} u_\ell^{(j)}}$$

$P(Y_{n+1} = j \mid Y_n = i)$ $\qquad$ $P(Y_{n+1} = \ell \mid Y_n = i)$

Example

Rate diagram



absorbing

Generator

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -9 & 1 & 2 & 4 & 2 \\ 3 & -5 & 2 & & \\ & 1 & -1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \end{array}$$

Compute $P(Y_M = 3)$ if $P(X_0 = i) = p_i$ for $i = 0,1,2$

$\sum p_i = 1$

Denote $u_i = P(Y_M = 3 \mid Y_0 = i)$.

$$\begin{cases} u_0 = \frac{1}{9} u_1 + \frac{2}{9} u_2 + \frac{4}{9} \\ u_1 = \frac{3}{5} u_0 + \frac{2}{5} u_2 \\ u_2 = u_1 \end{cases}$$

$$\begin{cases} u_0 = \frac{1}{9} u_0 + \frac{2}{9} u_0 + \frac{4}{9}, \quad u_0 = u_1 = u_2 = \frac{2}{3} \\ u_1 = u_0 \\ u_2 = u_1 \end{cases}$$

$$P(Y_M = 3) = \sum_{i=0}^{2} u_i \cdot P(X_0 = i) = \frac{2}{3} \sum_{i=0}^{2} p_i = \frac{2}{3}$$

Mean time to absorption

Similar analysis as was applied to B&D processes can be used to compute the mean time to absorption: before each jump from step i to state j the process sojourns $\frac{1}{q_i}$ on average in state i.

$$Q = \begin{array}{c} \\ 0 \\ \vdots \\ K-1 \\ K \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & \cdots & K-1 & K & \cdots & N \end{array} \\ \left(\begin{array}{ccc|ccc} -q_0 & & & & \cdots & q_{ij} \\ \vdots & & & & \vdots & \\ q_{ij} & \cdots & -q_{K-1} & & & \\ \hline & & & 0 & & 0 \\ & 0 & & & \ddots & \\ & & & & & 0 \end{array}\right) \end{array}$$

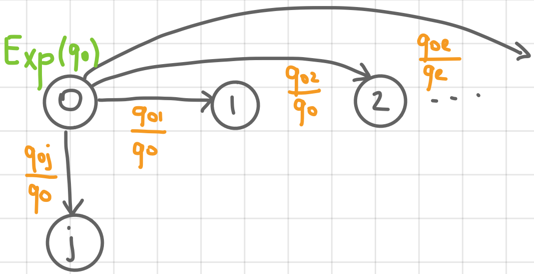
Let $T = \min\{t : X_t \in \{K, \dots, N\}\}$

$M = \min\{n : Y_n \in \{K, \dots, N\}\}$

Denote $w_i = E(T \mid X_0 = i)$

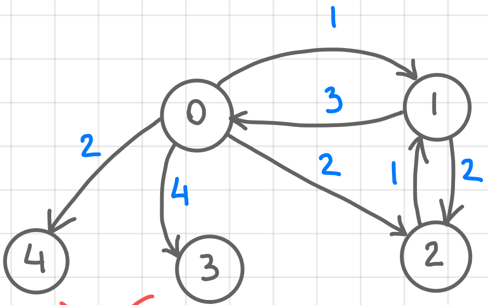
Then FSA gives

$$\boxed{w_i = \frac{1}{q_i} + \sum_{\substack{\ell=0 \\ \ell \neq i}}^{K-1} w_\ell \frac{q_{i\ell}}{q_i}}$$



Example

Rate diagram



absorbing

Generator

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -9 & 1 & 2 & 4 & 2 \\ 3 & -5 & 2 & & \\ & 1 & -1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \end{array}$$

$$T = \min\{t : X_t \in \{3,4\}\}$$

$$w_i = E(T \mid X_0 = i)$$

$$\begin{cases} w_0 = \frac{1}{9} + \frac{1}{9} w_1 + \frac{2}{9} w_2 \\ w_1 = \frac{1}{5} + \frac{3}{5} w_0 + \frac{2}{5} w_2 \\ w_2 = 1 + 1 \cdot w_1 \end{cases}$$

$$w_2 = 1 + w_1$$

$$w_0 = 1$$

$$w_1 = 2$$

$$w_2 = 3$$

Kolmogorov equations

Jump and hold description is very intuitive, gives a very clear picture of the process, but does not answer to some very basic questions, e.g., computing $P_{ij}(t) := P(X_t = j \mid X_0 = i)$.

For computing the transition probabilities the differential equation approach is more appropriate.

In order to derive the system of differential equations for $P_{ij}(t)$ from the infinitesimal description, we start from the familiar relation:

Chapman - Kolmogorov equation (semigroup property)

Chapman-Kolmogorov equation

$$P_{ij}(t+s) = P(X_{t+s}=j \mid X_0=i)$$

condition on the value of X_t

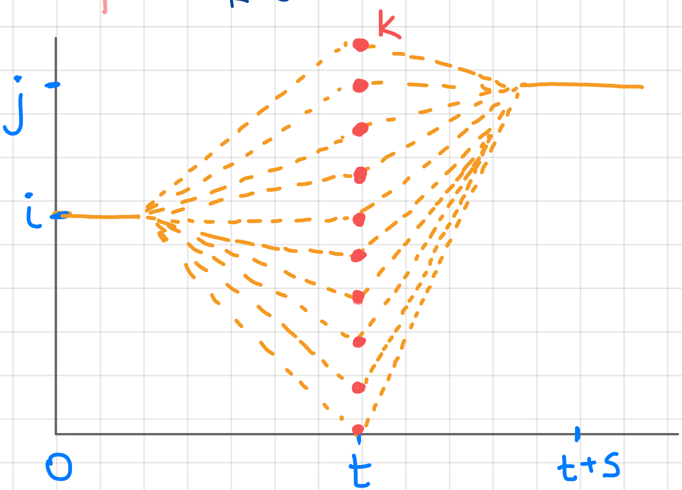
$$= \sum_{k=0}^{N} P(X_{t+s}=j \mid X_0=i, X_t=k)\, P(X_t=k \mid X_0=i)$$

Markov

$$= \sum_{k=0}^{N} P(X_{t+s}=j \mid X_t=k)\, P(X_t=k \mid X_0=i)$$

stationary trans. prob.

$$= \sum_{k=0}^{N} P(X_s=j \mid X_0=k)\, P(X_t=k \mid X_0=i) = \sum_{k=0}^{N} P_{ik}(t)\, P_{kj}(s)$$



Or in matrix form

$$P(t+s) = P(t)\,P(s)$$

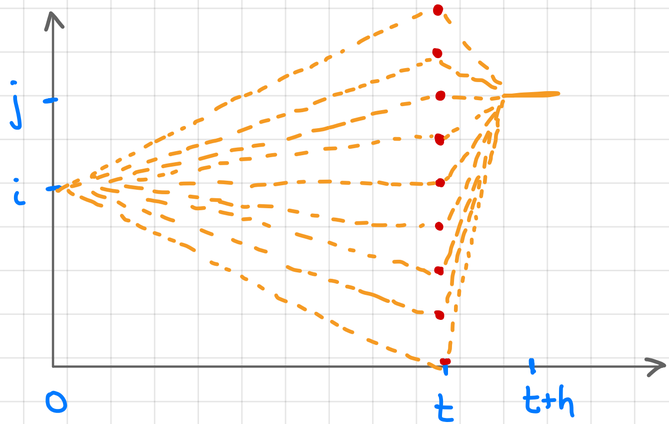
Kolmogorov forward equations

$Q = (q_{ij})_{i,j=0}^{N}$

Apply Chapman-Kolmogorov equations to compute

$P_{ij}(t+h)$:

$$P_{ij}(t+h) = \sum_{k=0}^{N} P_{ik}(t)\, P_{kj}(h) \qquad (*)$$



Use infinitesimal description:

$$P_{kj}(h) = \begin{cases} q_{kj}\, h + o(h), & k \neq j \\ 1 + q_{jj}\, h + o(h), & k = j \end{cases}$$

$$(*) = P_{ij}(t)\left(1 + q_{jj}\, h + o(h)\right) + \sum_{\substack{k=0 \\ k\neq j}}^{N} P_{ik}(t)\left(q_{kj}\, h + o(h)\right)$$

$$= P_{ij}(t) + \underbrace{\sum_{k=0}^{N} P_{ik}(t)\, q_{kj}}_{[P(t)Q]_{ij}} \cdot h + o(h)$$

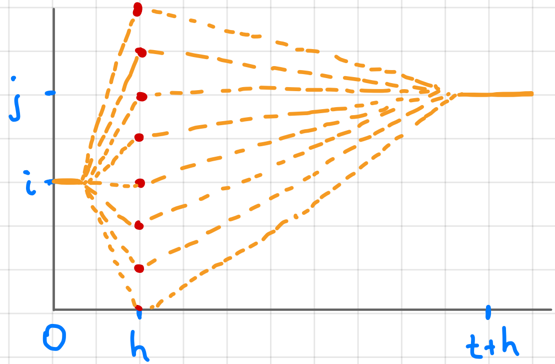
$$\boxed{\frac{d}{dt} P(t) = P(t)\, Q}$$

Kolmogorov backward equations

$$P_{ij}(t+h) = \sum_{k=0}^{N} P_{ik}(h) P_{kj}(t)$$

$$= \left(1 + q_{ii} h + o(h)\right) P_{ij}(t)$$

$$+ \sum_{\substack{k=0 \\ k \neq i}}^{N} \left(q_{ik} h + o(h)\right) P_{kj}$$



$$= P_{ij}(t) + \sum_{k=0}^{N} q_{ik} P_{kj}(t) h + o(h)$$

$$\hookrightarrow \boxed{\frac{d}{dt} P(t) = Q P(t)}$$

$$P(0) = I$$

Kolmogorov equations. Remarks

1. e^{tQ} satisfies both (forward and backward) equations.

Indeed, omitting technical details, differentiate term-by-term

$$\frac{d}{dt} e^{tQ} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{Q^k t^k}{k!} \right) = \sum_{k=1}^{\infty} \frac{Q^k t^{k-1}}{(k-1)!} = \sum_{\ell=0}^{\infty} \frac{Q^{\ell+1} t^{\ell}}{\ell!} = Q \sum_{\ell=0} \frac{Q^{\ell} t^{\ell}}{\ell!}$$

Now $\displaystyle\sum_{k=1}^{\infty} \frac{Q^k}{(k-1)!} t^{k-1} \overset{\ell=k-1}{=} \sum_{\ell=0}^{\infty} \frac{Q^{\ell+1}}{\ell!} t^{\ell} = Q e^{tQ} = e^{tQ} Q$ $\displaystyle = \left(\sum_{\ell=0}^{\infty} \frac{Q^{\ell} t^{\ell}}{\ell!} \right) Q$

2. Redundancy is related to the stationarity of transition probabilities. If transition probabilities $P_{ij}(s,t) = P(X_t = j \mid X_s = i)$ are not stationary, then

$\frac{\partial}{\partial t} P_{ij}(s,t) \to$ forward equation , $\frac{\partial}{\partial s} P_{ij}(s,t) \to$ backward equation

Example

Two-state MC $\qquad Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$

$$Q^2 = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} = \begin{pmatrix} \alpha(\alpha+\beta) & -\alpha(\alpha+\beta) \\ -\beta(\alpha+\beta) & \beta(\alpha+\beta) \end{pmatrix} = -(\alpha+\beta)\,Q$$

$\hookrightarrow \quad Q^k = (-1)^{k-1}(\alpha+\beta)^{k-1} Q \,, \qquad k \geq 1$

$$e^{tQ} = \sum_{k=0}^{\infty} \frac{Q^k t^k}{k!} = I + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\alpha+\beta)^{k-1} t^k}{k!} Q$$

$$= I - \frac{1}{\alpha+\beta} \sum_{k=1}^{\infty} \frac{(-(\alpha+\beta))^k t^k}{k!} Q$$

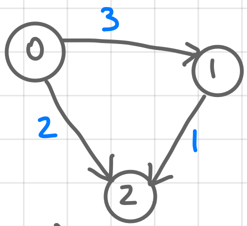
$$= I - \frac{1}{\alpha+\beta}\left(e^{-(\alpha+\beta)t} - 1\right) Q$$

$$= I + \frac{1}{\alpha+\beta} Q - \frac{1}{\alpha+\beta} e^{-(\alpha+\beta)t} Q$$

Example

Let $(X_t)_{t\geq 0}$ be a MC with generator Q

$$Q = \begin{pmatrix} -5 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



Compute $P_{01}(t)$

For any k, $Q^k = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$, $\Rightarrow P_{10}(t) = P_{20}(t) = P_{21}(t) = 0$

$$P'(t) = \begin{pmatrix} P_{00} & P_{01} & P_{02} \\ 0 & P_{11} & P_{12} \\ 0 & 0 & P_{22} \end{pmatrix} \begin{pmatrix} -5 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$P_{00}'(t) = -5 P_{00}(t)$, $P_{00}(0) = 1 \Rightarrow P_{00}(t) = e^{-5t}$

$P_{11}'(t) = -P_{11}(t)$, $P_{11}(0) = 1 \Rightarrow P_{11}(t) = e^{-t}$

$P_{22}'(t) = 0$, $P_{22}(0) = 1 \Rightarrow P_{22}(t) = 1$

$P_{01}'(t) = 3P_{00}(t) - P_{01}(t)$

$P_{01}(0) = 0$

$P_{01}'(t) =$ $\vdots$

$P_{01}(t) =$ $\vdots$

$$P_{01}(t) = \frac{3}{4}\left(e^{-t} - e^{-5t}\right) = \frac{3}{5} \cdot \frac{5}{4}\left(e^{-t} - e^{-5t}\right)$$

Forward and backward equations for B&D processes

Forward equation:

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(h)$$

$$= P_{ij-1}(t)\left(\lambda_{j-1} h + o(h)\right) + P_{ij}(t)\left(1-(\lambda_j+\mu_j)h + o(h)\right)$$

$$+ P_{ij+1}(t)\left(\mu_{j+1} h + o(h)\right) \quad + \theta_{ij}$$

If $\theta_{ij} = o(h)$ (requires additional technical assumptions)

$$\begin{cases} P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j+\mu_j) P_{ij}(t) + \mu_{j+1} P_{i,j+1}(t) \\ P_{i0}'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \end{cases} , \quad \text{with} \quad P_{ij}(0) = \delta_{ij}$$

Forward and backward equations for B&D processes

Similarly, we derive the backward equations

$P_n(t) = P(X_t = n)$

$$\begin{cases} P_{ij}'(t) = \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) + \lambda_i P_{i+1,j}(t) \\ P_{0j}(t) = -\lambda_0 P_{0j}(t) - \lambda_0 P_{1j}(t) \end{cases} \quad , \quad \text{with } P_{ij}(0) = \delta_{ij}$$

<u>Example</u> Linear growth with immigration.

Recall

$$\lambda_k = \underbrace{\lambda \cdot k}_{\text{linear birth rate}} + \underbrace{a}_{\text{immigration}}$$

$$\mu_k = \underbrace{\mu \cdot k}_{\text{linear death rate}}$$

Example: Linear growth with immigration.

Use forward equations to compute $E(X_t \mid X_0 = i)$

$$\begin{cases} P_{ij}'(t) = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{ij}(t) + \mu_{j+1} P_{i,j+1}(t) \\ P_{i0}'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \end{cases}$$

$$M'(t) = \sum_{j=0}^{\infty} j \cdot P_{ij}'(t)$$

$$E(X_t \mid X_0 = i) = \sum_{j=0}^{\infty} j \cdot P(X_t = j \mid X_0 = i) = \sum_{j=0}^{\infty} j \cdot P_{ij}(t) =: M(t)$$

$$j \,\Big|\, P_{ij}'(t) = (\lambda(j-1) + a) P_{i,j-1}(t) - ((\lambda+\mu)j + a) P_{ij}(t) + \mu(j+1) P_{i,j+1}(t)$$

$$(k+1) P_{i,k+1}'(t) \xrightarrow{\quad P_{ik}(t) \quad} (\cancel{k}+1)(\lambda k + a) P_{ik}(t)$$

$$k P_{i,k}'(t) \longrightarrow -k((\cancel{\lambda}+\cancel{\mu})k + \cancel{a}) P_{ik}(t)$$

$$(k-1) P_{i,k-1}'(t) \longrightarrow (\cancel{k}-1)\mu k P_{i,k}(t)$$

$$(\lambda k - \mu k + a) P_{ik}(t)$$

Example: Linear growth with immigration.

$$M'(t) = \sum_{j=0}^{\infty} j P_{ij}'(t) = \sum_{k=0}^{\infty} (\lambda-\mu) k P_{ik}(t) + \sum_{k=0}^{\infty} a P_{ik}(t)$$

$$= (\lambda-\mu) \sum_{k=0}^{\infty} k P_{ik}(t) + a \sum_{k=0}^{\infty} P_{ik}(t)$$

$$= (\lambda-\mu) M(t) + a$$

$$\begin{cases} M'(t) = (\lambda-\mu) M(t) + a \\ M(0) = i \end{cases}$$

$$\begin{cases} M(t) = i + at & \text{if } \lambda = \mu \\ M(t) = \dfrac{a}{\lambda-\mu}\left(e^{(\lambda-\mu)t} - 1\right) + i e^{(\lambda-\mu)t} & \text{if } \lambda \neq \mu \end{cases}$$