Name (last, first): ______________________________

Student ID: ______________________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

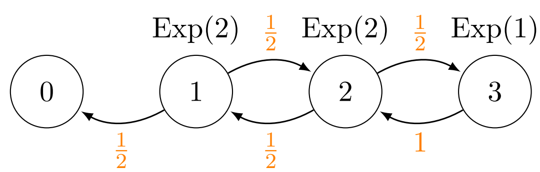
☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use two 8.5 by 11 inch sheets of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. (15 points) Let $(X_t)_{t\geq 0}$ be a birth and death process on states $\{0,1,2,3\}$ with state 0 absorbing, birth rates $\lambda_1 = 1$, $\lambda_2 = 1$ and the death rates $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1$.
   (a) (5 points) Draw the diagram of the jump chain of $(X_t)_{t\geq 0}$, indicate the distribution of the sojourn times. Is $(X_t)_{t\geq 0}$ irreducible?
   (b) (10 points) Suppose that X_0, the state of the process at time $t = 0$, is uniformly distributed on the set $\{1,2,3\}$. Compute the expectation of the time at which the process is absorbed in state 0.

Solution.

(a) The diagram of the jump chain of $(X_t)_{t\geq 0}$ has the following form



State 0 is absorbing. If $(Y_n)_{n\geq 0}$ is the embedded jump chain for $(X_t)_{t\geq 0}$, then $P(Y_n \in \{1,2,3\}|Y_0 = 0) = 0$ for all $n > 0$. Thus $(X_t)_{t\geq 0}$ is not irreducible.

(b) Denote by v_i the expected time to absorption given that $X_0 = i$, $i \in \{1,2,3\}$. Then, using the first step analysis, v_1, v_2, v_3 satisfy the following system of equations

$$
\begin{aligned}
v_1 &= \frac{1}{2} + \frac{1}{2}v_2,\\
v_2 &= \frac{1}{2} + \frac{1}{2}v_1 + \frac{1}{2}v_3,\\
v_3 &= 1 + v_2.
\end{aligned}
$$

Substituting the first and the third equations into the second, we get

$$
\begin{aligned}
v_2 &= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}v_2\right) + \frac{1}{2}(1 + v_2),\\
v_2 &= \frac{5}{4} + \frac{3}{4}v_2,\\
v_2 &= 5, \quad v_3 = 6, \quad v_1 = 3.
\end{aligned}
$$

Using the law of total probability, the average time to absorption at state 0 is equal to

$$\frac{1}{3}v_1 + \frac{1}{3}v_2 + \frac{1}{3}v_3 = \frac{1}{3}(3+5+6) = \frac{14}{3}.$$

2. (15 points) Let $(X_t)_{t\geq 0}$ be a birth and death process on states $\{0, 1, 2, 3\}$ with birth rates $\lambda_0 = 1$, $\lambda_1 = 1$, $\lambda_2 = 3$ and the death rates $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1$.

   (a) (5 points) Determine the infinitesimal generator of $(X_t)_{t\geq 0}$. Explain why $(X_t)_{t\geq 0}$ is irreducible.

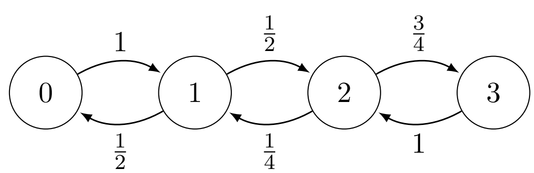
   (b) (10 points) Compute the stationary distribution for $(X_t)_{t\geq 0}$. [Hint. Remember that there are different ways of finding the stationary distribution]. What is the average fraction of time that the process spends in state 3 in the long run?

Solution.

(a) The infinitesimal generator of $(X_t)_{t\geq 0}$ is given by

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The diagram of the jump chain is



All states communicate, therefore the jump chain is irreducible. This implies that $(X_t)_{t\geq 0}$ is also irreducible.

(b) Denote by $(\pi_0, \pi_1, \pi_2, \pi_3)$ the stationary (limiting) distribution. Write the detailed balance equations for $(\pi_0, \pi_1, \pi_2, \pi_3)$

$$\pi_0 = \pi_1 \tag{1}$$
$$\pi_1 = \pi_2 \tag{2}$$
$$3\pi_2 = \pi_3. \tag{3}$$

Express π_1, π_2 and π_3 in terms of π_0

$$\pi_1 = \pi_0, \quad \pi_2 = \pi_0, \quad \pi_3 = 3\pi_0.$$

Substitute this into the equation $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$ gives $6\pi_0 = 1$, so

$$\pi_0 = \frac{1}{6}, \quad \pi_1 = \frac{1}{6}, \quad \pi_2 = \frac{1}{6}, \quad \pi_3 = \frac{1}{2}.$$

Therefore, in the long run, the process will spend $\pi_3 = \frac{1}{2}$ of time in states 3.

3. (15 points) Let X and Y be random variables. Suppose that $X \sim \mathrm{Unif}\big([0,10]\big)$, and given $X = x$, Y is distributed on $[0,x]$ with quadratic density

$$f_{Y|X}(y|x) = \alpha_x y^2.$$

(a) (5 points) Determine α_x.

(b) (5 points) Compute $E(Y|X = x)$.

(c) (5 points) Compute $E(Y)$.

(a) In order to determine α_x we use that $\int_0^x \alpha_x y^2 dy = 1$ for all $x \in [0,10]$

$$\int_0^x \alpha_x y^2 dy = \alpha_x \frac{x^3}{3} = 1 \quad \Rightarrow \quad \alpha_x = \frac{3}{x^3}.$$

(b) Compute the conditional expectation

$$E(Y|X = x) = \int_0^x y f_{Y|X}(y|x) dy = \int_0^x \frac{3}{x^3} y^3 dy = \frac{3}{x^3}\frac{x^4}{4} = \frac{3}{4}x.$$

(c) Now we compute $E(Y)$ by conditioning on X

$$E(Y) = \int_{-\infty}^{\infty} E(Y|X = x) f_X(x) dx = \int_0^{10} \frac{3}{4}x \cdot \frac{1}{10} dx = \frac{3}{40} \cdot \frac{100}{2} = \frac{15}{4}.$$

4. (15 points) John lives according to the following schedule. He starts his day by tossing a (fair) coin. He tosses the coin once every second until the first heads comes up. Once the first heads comes up, John turns on the TV and watches it for a random time with exponential distribution with parameter 5 (in seconds). Then he immediately turns the TV off, goes back to tossing the coin until the first heads comes up, and the whole process repeats anew.

   In the long run, what is the probability that at a given moment during the day John is watching TV? [To get the full credit, you have to model John's behavior as a stochastic process and clearly define all necessary objects and parameters related to this process before using any results from the lectures.]

Solution. Denote by $S(t)$ the process that describes John's activity at time t with

$$S(t) = \begin{cases} 0 & \text{if John tosses the coin at time } t, \\ 1 & \text{if John watches TV at time } t, \end{cases}$$

where $t = 0$ corresponds to the time when John starts his day. Then $S(0) = 0$ and the (random) moments of time when $S(t)$ switches from state 1 to state 0 form a renewal process. Denote this process $(X_t)_{t\geq 0}$.

Let $X_1, X_2, \ldots$ be i.i.d. random variables with geometric distribution with parameter $1/2$, and let $Y_1, Y_2, \ldots$ be i.i.d. random variables with exponential distribution with parameter 5 independent of $(X_i)_{i\geq 1}$. Random variables X_i represent the intervals of time during which John tosses the coin. Random variables Y_i represent the intervals of time during which John watches TV.

Then $(X_t)_{t\geq 0}$ is a two component renewal process with interrenewal times $Z_i = X_i + Y_i$. The random variables X_i represents the time when $S(t) = 0$ with $0 \leq X_i \leq Z_i$.

The interrenewal times have finite expectations

$$E(Z_i) = E(X_i) + E(Y_i) = 2 + \frac{1}{5} = \frac{11}{5} < \infty.$$

Therefore, it follows from the theorem about two component renewals (Lecture 20, page 7) that

$$\lim_{t\to\infty} P(S(t) = 0) = \frac{E(X_1)}{E(Z_1)} = \frac{10}{11}.$$

We conclude that

$$\lim_{t\to\infty} P(\text{John watches TV at time } t) = \lim_{t\to\infty} P(S(t) = 1) = 1 - \lim_{t\to\infty} P(S(t) = 0) = \frac{1}{11}.$$

5. (15 points) Let $\xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed random variables satisfying

$$P(\xi_i = 1) = p, \qquad P(\xi_i = -1) = 1 - p.$$

Consider a discrete-time stochastic process $(X_n)_{n\geq 0}$ with

$$X_0 = 1, \quad X_n = \beta^{S_n},$$

where $S_n = \sum_{i=1}^n \xi_i$, and $\beta > 0$ is a positive number.

(a) (5 points) Compute $E(\beta^{\xi_i})$.

(b) (10 points) Determine all values $\beta > 0$ for which X_n is a martingale.

Solutions.

(a) First, we compute $E(\beta^{\xi_i})$ for $\beta > 0$

$$E(\beta^{\xi_i}) = \frac{1-p}{\beta} + p\,\beta.$$

(b) If $(X_n)_{n\geq 0}$ is a martingale, then for any $n \geq 0$

$$E(X_{n+1}|X_0,\ldots,X_n) = E(\beta^{\xi_{n+1}}X_n|X_0,\ldots,X_n) = E(\beta^{\xi_{n+1}})X_n = X_n.$$

Notice that for any $n \geq 0$

$$\beta^{-n} \leq X_n \leq \beta^n.$$

Therefore, if $(X_n)_{n\geq 0}$ is a martingale, then $E(\beta^{\xi_i}) = 1$ for all $i \in \mathbb{N}$.
On the other hand, if $E(\beta^{\xi_i}) = 1$ for all $i \in \mathbb{N}$, then $E(X_n) = 1$ for all $n \in \mathbb{N}$, and $(X_n)_{n\geq 0}$ is a (multiplicative) martingale (as shown in lecture 21). We conclude that $(X_n)_{n\geq 0}$ is a martingale if and only if $E(\beta^{\xi_1}) = 1$.
Now we find all $\beta > 0$ such that $E(\beta^{\xi_1}) = 1$. Using part (a) we see that this is equivalent to finding all $\beta > 0$ such that

$$\frac{1-p}{\beta} + p\,\beta = 1.$$

If $p = 0$ or $p = 1$, then this equation has a unique solution $\beta = 1$. Suppose that $p \in (0,1)$. Then the above equation can be rewritten as

$$\beta^2 - \frac{1}{p}\beta + \frac{1-p}{p} = 0$$

with two solutions $\beta = 1$ and $\beta = \frac{1-p}{p}$.
The solution $\beta = 1$ gives a (trivial) case $X_n = 1$ for all $n \geq 0$.
For $p \in (0,1)$ the solution $\beta = \frac{1-p}{p}$ gives

$$X_n = \Big(\frac{1-p}{p}\Big)^{S_n},$$

which is a multiplicative martingale.

6. (15 points) The market price of a share is modeled by the Brownian motion with variance parameter $\sigma^2 = 2$ reflected at 0 (taking only positive values). Suppose that initially (at time $t = 0$) the price of the share is equal to 10.

   Determine the probability that at time $t = 50$ the price of the share is **greater than** 10, i.e., greater than the initial price. [Express the answer in terms of the CDF of the standard normal distribution $\Phi(x)$.]

   Solution. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, and let $(R_t)_{t\geq 0}$ be the price of the share at time t.

   It is given that $(R_t)_{t\geq}$ is the Brownian motion with variance $\sigma^2 = 2$ reflected at 0, i.e.,

$$R_t = |10 + \sqrt{2}B_t|.$$

   Therefore,

$$\begin{aligned}
P(R_{50} > 10) &= P(|10 + \sqrt{2}B_{50}| > 10) \\
&= P(|10 + 10B_1| > 10) \\
&= P(10 + 10B_1 > 10) + P(10 + 10B_1 < -10) \\
&= P(B_1 > 0) + P(B_1 < -2) \\
&= \frac{1}{2} + (1 - P(B_1 < 2)) \\
&= \frac{1}{2} + 1 - \Phi(2),
\end{aligned}$$

   where we used that B_1 has standard normal distribution.

7. (15 points) Let $(B_t^0)_{t\geq 0}$ be a Brownian bridge.
   (a) (5 points) Compute $P(|B_{1/2}^0| \leq 1)$. [You can leave your answer in terms of $\Phi(x)$, the CDF of the standard normal distribution.]
   (b) (10 points) Fix real numbers $\alpha_1, \ldots, \alpha_n$ and $0 < t_1 < \cdots < t_n < 1$. Determine the distribution of the random variable Z, where

$$Z := \sum_{i=1}^{n} \alpha_i B_{t_i}^0.$$

Solution.

(a) $(B_t^0)_{1\leq t\leq 1}$ is a Gaussian process with zero mean and covariance function $\Gamma(s,t) = \min\{s,t\} - st$. Thus,

$$\mathrm{Var}(B_{1/2}^0) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

and

$$B_{1/2}^0 \sim N\Big(0, \frac{1}{4}\Big).$$

Therefore,

$$P(|B_{1/2}^0| \leq 1) = P\Big(-1 \leq \frac{1}{2}B_1 \leq 1\Big) = P\Big(-2 \leq B_1 \leq 2\Big) = \Phi(2) - \Phi(-2) = 2\Phi(2) - 1,$$

where $B_1 \sim N(0,1)$.

(b) By using again that $(B_t^0)_{1\leq t\leq 1}$ is a Gaussian process with zero mean and covariance function $\Gamma(s,t) = \min\{s,t\} - st$ we find that Z has normal distribution with zero mean. It remains to compute the variance

$$\begin{aligned}
\mathrm{Var}(Z) &= \mathrm{Cov}(Z,Z) \\
&= \mathrm{Cov}\Big(\sum_{i=1}^{n} \alpha_i B_{t_i}^0, \sum_{j=1}^{n} \alpha_j B_{t_j}^0\Big) \\
&= \sum_{i,j=1}^{n} \alpha_i\alpha_j \mathrm{Cov}\Big(B_{t_i}^0, B_{t_j}^0\Big) \\
&= \sum_{i,j=1}^{n} \alpha_i\alpha_j \Big(\min\{t_i,t_j\} - t_it_j\Big).
\end{aligned}$$

We conclude that

$$Z \sim N\Big(0, \sum_{i,j=1}^{n} \alpha_i\alpha_j \Big(\min\{t_i,t_j\} - t_it_j\Big)\Big).$$

8. (15 points) Let $(X_t)_{t\geq 0}$ be a pure birth process of rates $(\lambda_k)_{k=0}^{\infty}$. We say that $(X_t)_{t\geq 0}$ *explodes* if the process $(X_t)_{t\geq 0}$ makes infinitely many jumps in a finite time interval.

   (a) (5 points) Let W_n be the nth waiting time of $(X_t)_{t\geq 0}$. Compute $E(e^{-W_n})$.

   (b) (5 points) Prove that for any sequence of positive numbers $(\lambda_n)_{n=0}^{\infty}$

$$\prod_{n=0}^{\infty}\Big(1+\frac{1}{\lambda_n}\Big) \geq \sum_{n=0}^{\infty}\frac{1}{\lambda_n},$$

   where

$$\prod_{n=0}^{\infty}\Big(1+\frac{1}{\lambda_n}\Big) := \lim_{k\to\infty}\prod_{n=0}^{k}\Big(1+\frac{1}{\lambda_n}\Big).$$

   (c) (5 points) Denote $W_\infty := \lim_{n\to\infty} W_n \in (0,+\infty)\cup\{+\infty\}$. Use parts (a) and (b) to prove that

$$\text{if} \quad \sum_{n=0}^{\infty}\frac{1}{\lambda_n} = +\infty, \quad \text{then} \quad P(W_\infty = +\infty) = 1.$$

   [You **do not** have to justify switching the operations of taking the limit and taking the expectation.]

   Conclude from this the following implication

$$\text{if} \quad \sum_{n=0}^{\infty}\frac{1}{\lambda_n} = +\infty, \quad \text{then} \quad P\big((X_t)_{t\geq 0}\text{ explodes}\big) = 0.$$

Solution.

(a) Let $S_0, S_1, \dots$ be the sojourn times of $(X_t)_{t\geq 0}$. Then $S_0, S_1, \dots$ are independent, $S_i \sim \text{Exp}(\lambda_i)$ and $W_n = S_0 + \cdots + S_{n-1}$. Therefore,

$$E(e^{-Wn}) = E\Big(e^{-\sum_{i=0}^{n-1} S_i}\Big) = \prod_{i=0}^{n-1} E\Big(e^{-S_i}\Big). \tag{4}$$

For each $i \geq 0$ we have

$$E\Big(e^{-S_i}\Big) = \int_0^\infty e^{-x}\lambda_i e^{-\lambda_i x}dx = \int_0^\infty \lambda_i e^{-(1+\lambda_i)x}dx = \frac{\lambda_i}{1+\lambda_i},$$

which together with (4) gives

$$E(e^{-Wn}) = \prod_{i=0}^{n-1}\frac{\lambda_i}{1+\lambda_i}. \tag{5}$$

(b) First we show that for any $k \geq 0$

$$\prod_{n=0}^{k}\Big(1+\frac{1}{\lambda_n}\Big) \geq \sum_{n=0}^{k}\frac{1}{\lambda_n}. \tag{6}$$

We prove (6) by induction on k. For $k = 0$ we have

$$1 + \frac{1}{\lambda_0} \geq \frac{1}{\lambda_0}. \tag{7}$$

Suppose that

$$\prod_{n=0}^{k}\Big(1 + \frac{1}{\lambda_n}\Big) \geq \sum_{n=0}^{k} \frac{1}{\lambda_n}. \tag{8}$$

Then

$$\prod_{n=0}^{k+1}\Big(1 + \frac{1}{\lambda_n}\Big) = \Big(1 + \frac{1}{\lambda_{k+1}}\Big)\prod_{n=0}^{k}\Big(1 + \frac{1}{\lambda_n}\Big) \tag{9}$$

$$= \prod_{n=0}^{k}\Big(1 + \frac{1}{\lambda_n}\Big) + \frac{1}{\lambda_{k+1}}\prod_{n=0}^{k}\Big(1 + \frac{1}{\lambda_n}\Big) \tag{10}$$

$$\geq \sum_{n=0}^{k} \frac{1}{\lambda_n} + \frac{1}{\lambda_{k+1}} \tag{11}$$

$$= \sum_{n=0}^{k+1} \frac{1}{\lambda_n}, \tag{12}$$

where the inequality follows from (8) and $\prod_{n=0}^{k}\Big(1 + \frac{1}{\lambda_n}\Big) \geq 1$. This establishes the induction step. Using the principle of mathematical induction we conclude that (6) holds for all $k \geq 0$. Since both sequences $\prod_{n=0}^{k}\Big(1 + \frac{1}{\lambda_n}\Big)$ and $\sum_{n=0}^{k} \frac{1}{\lambda_n}$ are monotonically increasing, their limits are well defined, and we have

$$\prod_{n=0}^{\infty}\Big(1 + \frac{1}{\lambda_n}\Big) = \lim_{k\to\infty}\prod_{n=0}^{k}\Big(1 + \frac{1}{\lambda_n}\Big) \geq \lim_{k\to\infty}\sum_{n=0}^{k} \frac{1}{\lambda_n} = \sum_{n=0}^{\infty} \frac{1}{\lambda_n}. \tag{13}$$

(c) Suppose that $\sum_{i=0}^{\infty} \frac{1}{\lambda_i} = +\infty$. Then it follows from part (b) that

$$\prod_{i=0}^{\infty}\Big(1 + \frac{1}{\lambda_i}\Big) = \lim_{n\to\infty}\prod_{i=0}^{n}\Big(\frac{1+\lambda_i}{\lambda_i}\Big) = +\infty. \tag{14}$$

Combining this with equation (5) from part (a) we have

$$\lim_{n\to\infty} E(e^{-W_n}) = \lim_{n\to\infty}\prod_{i=0}^{n}\frac{\lambda_i}{1+\lambda_i} = \frac{1}{\lim_{n\to\infty}\prod_{i=0}^{n}\frac{1+\lambda_i}{\lambda_i}} = 0. \tag{15}$$

W_n is a non-decreasing sequence and e^{-W_n} are non-negative, therefore, by the monotone convergence theorem

$$\lim_{n\to\infty} E(e^{-W_n}) = E(\lim_{n\to\infty} e^{-W_n}) = E(e^{-W_\infty}) = 0. \tag{16}$$

(You do not need to justify switching between taking the limit and taking the expectation.) Since $e^{-W_\infty} \geq 0$ and $E(e^{-W_\infty}) = 0$, we conclude that

$$P(e^{-W_\infty} = 0) = 1, \tag{17}$$

which implies that

$$P(W_\infty = +\infty) = 1. \tag{18}$$

The event that $(X_t)_{t\geq 0}$ explodes is equivalent to $\{W_\infty < +\infty\}$. Therefore, if $\sum_{i=0}^{\infty} \frac{1}{\lambda_i} = +\infty$, then

$$P\big((X_t)_{t\geq 0} \text{ explodes}\big) = P(W_\infty < +\infty) = 1 - P(W_\infty = +\infty) = 0. \tag{19}$$