MATH 180C MIDTERM 1 4/28/23, 4-4:50 PM, CENTR 109

Name (last, first): ____________________

Student ID: ____________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use one 8.5 by 11 inch sheet of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. (40 points) Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain on the state space $\{0,1,2\}$ with transition probability functions

$$P(t) = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{Vmatrix} e^{-3t} & \frac{1}{2} - e^{-3t} + \frac{1}{2}e^{-4t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 0 & \frac{1}{2} + \frac{1}{2}e^{-4t} & \frac{1}{2} - \frac{1}{2}e^{-4t} \\ 0 & \frac{1}{2} - \frac{1}{2}e^{-4t} & \frac{1}{2} + \frac{1}{2}e^{-4t} \end{Vmatrix} \end{array}. \tag{1}$$

(a) Compute the generator Q of $(X_t)_{t\geq 0}$. [Hint. Recall that $P'(0) = Q$.]

(b) Give the jump-and-hold description of $(X_t)_{t\geq 0}$.

(c) Draw the rate diagram of $(X_t)_{t\geq 0}$.

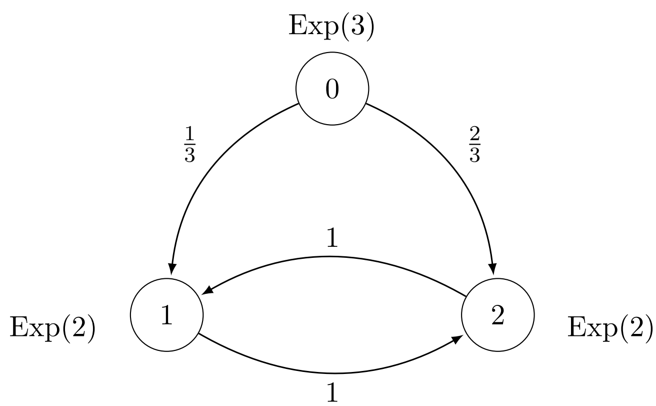
(d) Assuming that X_0 is uniformly distributed on $\{0,1,2\}$, compute the probability that $X_1 = 2$.

Solution.

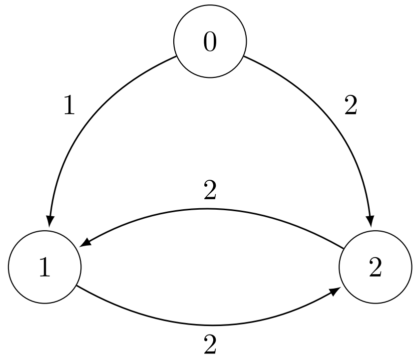
(a) Computing $P'(0)$ gives

$$Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{Vmatrix} -3 & 1 & 2 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{Vmatrix} \end{array} \tag{2}$$

(b) The jump-and-hold diagram



(c) The rate diagram



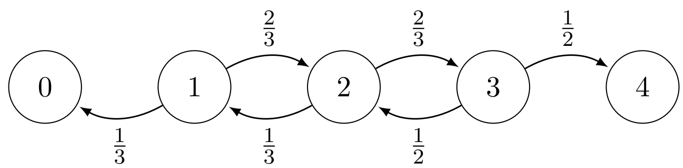
(d)

$$\begin{aligned} P(X_1 = 2) &= \sum_{i=0}^{2} P(X_1 = 2 \,|\, X_0 = i) P(X_0 = i) && (3) \\ &= \frac{1}{3}\Big(P_{02}(1) + P_{12}(1) + P_{22}(1)\Big) && (4) \\ &= \frac{1}{2} - \frac{1}{6}e^{-4}. && (5) \end{aligned}$$

2. (30 points) Let $(X_t)_{t\geq 0}$ be a birth and death process on states $\{0,1,2,3,4\}$ with states 0 and 4 absorbing, birth rates $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 1$ and death rates $\mu_1 = 1, \mu_2 = 1, \mu_3 = 1$.
   (a) Draw the diagram for the jump chain of $(X_t)_{t\geq 0}$.
   (b) Suppose that X_0, the state of the process at time $t = 0$, is uniformly distributed on $\{1,2,3\}$. What is the probability that the process is absorbed in state 0?

Solution.

(a) The diagram of the jump chain is



(b) We have the system of equations

$$u_1 = \frac{1}{3} + \frac{2}{3}u_2 \tag{6}$$
$$u_2 = \frac{1}{3}u_1 + \frac{2}{3}u_3 \tag{7}$$
$$u_3 = \frac{1}{2}u_2 \tag{8}$$

which gives

$$u_2 = \frac{1}{9} + \frac{2}{9}u_2 + \frac{1}{3}u_2 \tag{9}$$
$$\frac{4}{9}u_2 = \frac{1}{9} \tag{10}$$
$$u_2 = \frac{1}{4}, \quad u_1 = \frac{1}{3} + \frac{2}{3}\frac{1}{4} = \frac{6}{12} = \frac{1}{2}, \quad u_3 = \frac{1}{8} \tag{11}$$
$$\frac{1}{3}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{1}{3}\frac{1+2+4}{8} = \frac{7}{24} \tag{12}$$

3. (30 points) Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain on the state space $\{0,1,2\}$ with generator

$$Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} -2 & 2 & 0 \\ 3 & -6 & 3 \\ 0 & 1 & -1 \end{array} \right\| \end{array}. \tag{13}$$

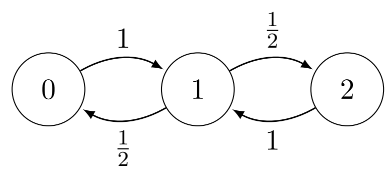
(a) Draw the diagram for the jump chain of $(X_t)_{t\geq 0}$ and explain why $(X_t)_{t\geq 0}$ is irreducible.

(b) Compute the stationary distribution for $(X_t)_{t\geq 0}$.

(c) What is the expected average fraction of time that $(X_t)_{t\geq 0}$ spends in states 1 and 2 in the long run?

Solution.

(a) The diagram of the jump chain is



All states communicate, therefore the jump chain is irreducible. This implies that $(X_t)_{t\geq 0}$ is also irreducible.

(b) Denote by (π_0, π_1, π_2) the limiting distribution. Then (π_0, π_1, π_2) satisfy the following system

$$-2\pi_0 + 3\pi_1 = 0, \tag{14}$$
$$2\pi_0 - 6\pi_1 + \pi_2 = 0, \tag{15}$$
$$3\pi_1 - \pi_2 = 0, \tag{16}$$
$$\pi_0 + \pi_1 + \pi_2 = 1. \tag{17}$$

The first equation gives $\pi_0 = \frac{3}{2}\pi_1$, the third equation gives $\pi_2 = 3\pi_1$. Plugging this into the last equation gives

$$\pi_1\left(\frac{3}{2} + 1 + 3\right) = \pi_1\frac{11}{2} = 1, \tag{18}$$

so

$$\pi_1 = \frac{2}{11}, \quad \pi_0 = \frac{3}{11}, \quad \pi_2 = \frac{6}{11}. \tag{19}$$

(c) In the long run, the process spends $\pi_1 + \pi_2 = \frac{8}{11}$ of time in states 1 and 2.