☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

1. (30 points) Let $(X_t)_{t\geq 0}$ be a birth and death process on states $\{0,1,2,3\}$ with state 0 absorbing, birth rates $\lambda_1 = 2$, $\lambda_2 = 1$ and the death rates $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1$.
   (a) Draw the diagram of the jump chain of $(X_t)_{t\geq 0}$ and indicate the distribution of the sojourn times.
   (b) Suppose that X_0, the state of the process at time $t = 0$, is uniformly distributed on the set $\{1,2,3\}$. Compute the expectation of the time at which the process is absorbed at state 0.

2. (30 points) Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain on the state space $\{0,1,2\}$ with transition probability functions

$$P(t) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c|c|c} 0 & 1 & 2 \\ \frac{1}{6}+\frac{3}{2}e^{-4t}-\frac{2}{3}e^{-3t} & \frac{1}{6}-\frac{3}{2}e^{-4t}+\frac{4}{3}e^{-3t} & \frac{2}{3}-\frac{2}{3}e^{-3t} \\ \frac{1}{6}+\frac{1}{2}e^{-4t}-\frac{2}{3}e^{-3t} & \frac{1}{6}-\frac{1}{2}e^{-4t}+\frac{4}{3}e^{-3t} & \frac{2}{3}-\frac{2}{3}e^{-3t} \\ \frac{1}{6}-\frac{1}{2}e^{-4t}+\frac{1}{3}e^{-3t} & \frac{1}{6}+\frac{1}{2}e^{-4t}-\frac{2}{3}e^{-3t} & \frac{2}{3}+\frac{1}{3}e^{-3t} \end{array}.$$

   (a) Determine the distribution of the sojourn times of the process at states 0, 1 and 2.
   (b) In the long run, what fraction of time will the process $(X_t)_{t\geq 0}$ spend in state 0? [**Hint.** You can answer this question without solving any equations, and if you do so you should clearly state which results you use.]
   (c) Let $Q = (q_{ij})_{i,j=0}^{2}$ be the generator matrix of $(X_t)_{t\geq 0}$. Compute q_{10}. Suppose you observe the process jumping from state 2 to state 0. What is the average time that you have to wait until the next time you observe the jump from state 2 to state 0?

3. (30 points) Certain printing facility has two printers operating on a 24/7 basis and one repairman that takes care of the printers. The amount of time (in hours) that a printer works before breaking down has exponential distribution with mean 2. If a printer is broken, the repairman needs exponentially distributed amount of time with mean 1 (hour) to repair the broken printer. The repairman cannot repair two printers simultaneously. Each printer can produce 100 pages per minute.

   Let X_t denote the number of printers in operating state at time t.
   (a) Assuming without proof that $(X_t)_{t\geq 0}$ is a Markov process, determine the generator of $(X_t)_{t\geq 0}$ (you can provide rigorous computations for only one entry of matrix Q.) [Hint. If $T \sim \text{Exp}(\gamma)$, then $P(T \leq h) = \gamma h + o(h)$ as $h \to 0$.]
   (b) Compute the stationary distribution for $(X_t)_{t\geq 0}$.
   (c) In the long run, how many pages does the facility produce on average per minute?