☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem.**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

☐ **You may assume that all transition probability functions are STATIONARY.**

1. (30 points) Let $Y > 0$ be a random variable having Gamma distribution with parameters 2 and λ, i.e., the p.d.f. of Y is given by

$$f_Y(y) = \lambda^2 y e^{-\lambda y}, \qquad y > 0, \tag{1}$$

and let $X \sim \text{Unif}[0, Y]$ be a random variable uniformly distributed on $[0, Y]$.

It is given that $E(X) = 1$.

(a) (10 points) Determine the unknown parameter λ. [Hint. Compute $E(X)$ with unknown parameter λ.]

(b) (10 points) Determine

$$P(X \leq t \,|\, Y = y) = \begin{cases} \qquad\quad , & t < 0, \\ \qquad\quad , & 0 \leq t < y, \\ \qquad\quad , & t \geq y. \end{cases} \tag{2}$$

for $y > 0$.

(c) (10 points) Using the results from (a) and (b), compute $P(X \leq t)$ and determine the marginal distribution of X.

2. (35 points) Certain device consists of two components, A and B. Whenever one of the components fails, the whole device is immideately replaced by a new one. Components A and B are the only components that can fail.

Suppose that the lifetimes of components A and B (in days) are independent random variables both having exponential distributions with rate λ. Let $N(t)$ be the renewal process counting the number of the replacements of the device on the time interval $[0, t]$.

(a) (10 points) Express the interrenewal times in terms of the lifetimes of components A and B (hint: this is **not** a sum) and compute the distribution of the interrenewal times.

(b) (15 points) Determine an asymptotic expression for the mean age of the device at time t in the long run.

(c) (10 points) What is the long run probability that the device will fail within next 24 hours?

3. (35 points) The climate of a certain tropical country is characterized by the alternating periods of rain and (sunny) periods without precipitations. Let $(X_i)_{i \geq 0}$ and $(Y_i)_{i \geq 1}$ be the random variables describing the lengths of the consecutive rainy and sunny periods of time correspondingly, and assume that $(X_i)_{i \geq 0}$ and $(Y_i)_{i \geq 0}$ are two independent families of *i.i.d.* continuous random variables.

We start the observation at the beginning of one of the rainy periods and count the number of times the weather changes from sunny to rainy. Suppose that

$$E(X_1) = \alpha, \quad \text{Var}(X_1) = 2\alpha^2, \quad E(Y_1) = \beta, \quad \text{Var}(Y_1) = 2\beta^2 \tag{3}$$

for some $\alpha > 0, \beta > 0$.

(a) (20 points) Determine an asymptotic expression (linear and constant terms) of the expected number of times the weather changes from sunny to rainy on the interval $[0, t]$ for $t \gg 1$.

(b) (15 points) What is the long run average fraction of time that the weather in this country is sunny?