MATH 180C PRACTICE FINAL

Name (last, first): ______________________________

Student ID: ______________________________

☐ **Write your name and PID on the top of EVERY PAGE.**

☐ **Write the solutions to each problem on separate pages. CLEARLY INDICATE on the top of each page the number of the corresponding problem. Different parts of the same problem can be written on the same page (for example, part (a) and part (b)).**

☐ **Remember this exam is graded by a human being. Write your solutions NEATLY AND COHERENTLY, or they risk not receiving full credit.**

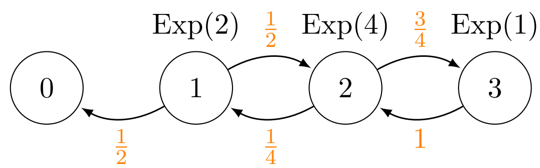
☐ **You may assume that all transition probability functions are STATIONARY.**

☐ **You are allowed to use two 8.5 by 11 inch sheets of paper with handwritten notes (on both sides); no other notes (or books) are allowed.**

1. Let $(X_t)_{t\geq 0}$ be a birth and death process on states $\{0,1,2,3\}$ with state 0 absorbing, birth rates $\lambda_1 = 1$, $\lambda_2 = 3$ and the death rates $\mu_1 = 1$, $\mu_2 = 1$, $\mu_3 = 1$.
   (a) Draw the diagram of the jump chain of $(X_t)_{t\geq 0}$ and indicate the distribution of the sojourn times.
   (b) Suppose that X_0, the state of the process at time $t = 0$, is uniformly distributed on the set $\{1,2,3\}$. Compute the expectation of the time at which the process is absorbed at state 0.

Solution.

(a) The diagram of the jump chain of $(X_t)_{t\geq 0}$ has the following form



The probability $P_{i,i+1}$ (the probability of jumping from state i to state $i+1$) is equal to

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i},$$

and, similarly,

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}.$$

The sojourn time at state i has exponential distribution with rate $\lambda_i + \mu_i$.

(b) Denote by v_i the expected time to absorption given that $X_0 = i$, $i \in \{1,2,3\}$. Then, using the first step analysis, v_1, v_2, v_3 satisfy the following system of equations

$$\begin{aligned} v_1 &= \frac{1}{2} + \frac{1}{2}v_2, \\ v_2 &= \frac{1}{4} + \frac{1}{4}v_1 + \frac{3}{4}v_3, \\ v_3 &= 1 + v_2. \end{aligned}$$

Substituting the first and the third equations into the second, we get

$$\begin{aligned} v_2 &= \frac{1}{4} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}v_2\right) + \frac{3}{4}(1 + v_2), \\ v_2 &= \frac{9}{8} + \frac{7}{8}v_2, \\ v_2 &= 9, \quad v_3 = 10, \quad v_1 = 5. \end{aligned}$$

Using the law of total probability, the average time to absorption at state 0 is equal to

$$\frac{1}{3}v_1 + \frac{1}{3}v_2 + \frac{1}{3}v_3 = \frac{1}{3}(5 + 9 + 10) = 8.$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

2. Certain device consists of two components. The amount of time that the components work before breaking down has exponential distribution with rate 1. If any of the components fails, the repair time has exponential distribution with mean 2. The two components work independently and are repaired independently of each other.

   The number of components working at time t is given by the process $(X_t)_{t\geq 0}$ which is a continuous time Markov chain.

   (a) Determine the genetator Q of $(X_t)_{t\geq 0}$.

   (b) Determine the stationary distribution of $(X_t)_{t\geq 0}$.

   (c) In the long run, what fraction of time both components work simultaneously?

Solution

(a) Let T_i and R_i denote the working and repair times of the ith component correspondingly for $i \in \{1, 2\}$. Then $T_i \sim \text{Exp}(1)$, $R_i \sim \text{Exp}(1/2)$ and the generator matrix Q of the Markov process $(X_t)_{t\geq 0}$ is given by (similarly as in Problem 4 of Practice Midterm 1)

$$Q = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} -1 & 1 & 0 \\ 1 & -3/2 & 1/2 \\ 0 & 2 & -2 \end{array} \right\| \end{array}. \tag{1}$$

For example, the (2,1) entry of Q can be computed from

$$\begin{aligned} P(X_h = 1 \,|\, X_0 = 2) &= P(\text{one of two working components fails during } (0,h)) + o(h) \qquad (2) \\ &= P(T_1 < h, T_2 > h) + P(T_2 < h, T_1 > h) + o(h) \qquad (3) \\ &= (1 - e^{-h})e^{-h} + (1 - e^{-h})e^{-h} + o(h) \qquad (4) \\ &= 2h + o(h) \qquad (5) \end{aligned}$$

as $h \to 0$, and we get that $q_{21} = 2$.

(b) Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the stationary distribution of the Markov chain $(X_t)_{t\geq 0}$. Then (see Lecture 10, page 5) π satisfies the following system of equations

$$\pi Q = 0, \qquad \pi_0 + \pi_1 + \pi_2 = 1. \tag{6}$$

Solve this system

$$\begin{aligned} -\pi_0 + \pi_1 &= 0, \qquad (7) \\ \pi_0 - \frac{3}{2}\pi_1 + 2\pi_2 &= 0, \qquad (8) \\ \frac{1}{2}\pi_1 - 2\pi_2 &= 0, \qquad (9) \\ \pi_0 + \pi_1 + \pi_2 &= 0. \qquad (10) \end{aligned}$$

From the first and the third equations we get that $\pi_0 = \pi_1 = 4\pi_2$, which together with the fourth equation gives

$$\pi_0 = \frac{4}{9}, \quad \pi_1 = \frac{4}{9}, \quad \pi_2 = \frac{1}{9}. \tag{11}$$

(c) The average long run fraction of time spent in state 2, corresponding to both components working simultaneously, is given by $\pi_2 = 1/9$ (see Lecture 10, page 11).

3. Let X and Y be random variables. Suppose that $X \sim \text{Exp}(2)$, and given $X = x$, Y is distributed on $[0, x]$ with linear density
$$f_{Y|X}(y|x) = \alpha_x y. \tag{12}$$
   (a) Determine α_x.
   (b) Compute $E(Y \mid X = x)$.
   (c) Compute $E(Y)$.

Solution.

(a) For each $x > 0$ function $\alpha_x y$ is a probability density on the interval $[0, x]$. Therefore, α_x satisfies
$$\int_0^x \alpha_x y dy = \alpha_x \frac{x^2}{2} = 1, \tag{13}$$
from which we conclude that $\alpha_x = \frac{2}{x^2}$.

(b) By definition,
$$E(Y \mid X = x) = \int_{-\infty}^{+\infty} y \cdot f_{Y|X}(y|x) dy = \int_0^x y \cdot \frac{2y}{x^2} dy = \frac{2}{3} x \tag{14}$$

(c) Finally, by conditioning on the value of X and using that $X \sim \text{Exp}(2)$, we get
$$E(Y) = \int_0^\infty E(Y|X = x) 2e^{-2x} dx = \int_0^\infty \frac{2}{3} x \cdot 2e^{-2x} dx = \frac{2}{3} E(X) = \frac{1}{3}. \tag{15}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

4. The economic history of a certain county is characterized by alternating periods of long economic growth and periods of long recession. Suppose that the lengths of all periods are independent and have uniform distribution on $[0,1]$ (in years), both for growth and recession. At the beginning of our observation (time $t=0$) a new recession starts.

   (a) Let X and Y be independent random variables having uniform distributions on [0,1]. Compute

$$P(X+Y \le t) = \begin{cases} & t \le 0, \\ & 0 < t \le 1, \\ & 1 < t \le 2, \\ & t > 2. \end{cases}$$

   [Hint. Draw a unit square.]

   (b) What is the long-run probability that there will be no new recession starting within next year [Hint. Formulate using the excess life.]

Solution.

(a)

$$P(X+Y \le t) = \begin{cases} 0, & t \le 0, \\ t^2/2, & 0 < t \le 1, \\ 1-(2-t)^2/2, & 1 < t \le 2, \\ 1, & t > 2. \end{cases} \tag{16}$$

(b) Let $N(t)$ count the number of times the economy switches from growth to recession on the time interval $[0,t]$. Then $N(t)$ is a renewal process with interrennewal times being the sum of two independent uniformly distributed on $[0,1]$ random variables (periods of recession and growth). The interrenewal distribution $F(t)$ is then equal to function (20) computed in part (a)

$$F(t) = \begin{cases} 0, & t \le 0, \\ t^2/2, & 0 < t \le 1, \\ 1-(2-t)^2/2, & 1 < t \le 2, \\ 1, & t > 2. \end{cases} \tag{17}$$

Each renewal denotes the start of a new recession. For any fixed time t, the event that there will be no new recession starting during the next year means in terms of the renewal process that the excess life at time t is greater than 1, i.e., $\gamma_t > 1$. If we observe the situation in the country at some time t with $t \gg 1$ large, then from the theorem about the limiting distribution of the excess life (Lecture 17, page 3)

$$P(\gamma_t > 1) = \frac{1}{\mu}\int_1^\infty (1-F(x))dx, \tag{18}$$

where μ is the mean interrenewal time. Plugging (20) and $\mu = 1$ into the above formula, we get

$$P(\gamma_t > 1) = \frac{1}{\mu}\int_1^\infty (1-F(x))dx = \int_1^2 \frac{(2-x)^2}{2}dx = \frac{1}{6}. \tag{19}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

5. Let $X_1, X_2, \ldots$ be *i.i.d.* random variables having exponential distribution with rate 1, i.e., $X_1 \sim \text{Exp}(1)$.

   (a) Let Y be an exponential random variable with rate λ. Compute

   $$M_Y(t) = E(e^{tY}) = \begin{cases} \quad , & t < \lambda, \\ \quad , & t \geq \lambda \end{cases} \tag{20}$$

   for $t \in (-\infty, \infty)$. (Recall that $M_Y(t)$ is called the moment generating function of Y).

   (b) Using the result from (a) show that for any $t < 1$, the process $(Z_n)_{n \geq 1}$ defined by

   $$Z_n = (1-t)^n e^{t \sum_{i=1}^n X_i}, \quad Z_0 = 1 \tag{21}$$

   is a nonegative martingale.

Solution.

(a) The moment generating function of Y is given by

$$E(e^{tY}) = \int_0^\infty e^{ty} \lambda e^{-\lambda y} dy = \int_0^\infty \lambda e^{-(\lambda - t)y} = \begin{cases} \frac{\lambda}{\lambda - t}, & t < \lambda, \\ +\infty, & t \geq \lambda. \end{cases} \tag{22}$$

(b) Using the result of part (a) with $\lambda = 1$, for $t < 1$

$$E((1-t)e^{tX_i}) = 1. \tag{23}$$

Therefore,

$$Z_n = \prod_{i=1}^n (1-t) e^{tX_i} \tag{24}$$

is a product of i.i.d. nonnegative random variables with mean 1, which defines a (nonnegative) multiplicative martingale.

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

6. Let $(X_t)_{t\geq 0}$ be a Brownian motion with drift μ and variance parameter σ^2. It is given that $X_0 = 0$, $E(X_1) = 1$ and $\operatorname{Var}(X_1) = 1$.

   (a) Determine μ and σ^2.

   (b) Suppose that the price fluctuations of a share are modeled by the process $(Z_t)_{t\geq 0}$ given by

   $$Z_t = e^{X_t}. \tag{25}$$

   Determine the probability that the price of the share doubles before it drops by one half (i.e., probability that the price increases from 1 to 2 before in drops from 1 to 1/2).

Solution.

(a) If $(X_t)_{t\geq 0}$ is a Brownian motion with drift μ and variance σ^2, then $E(X_t) = \mu t$ and $\operatorname{Var}(X_t) = \sigma^2$, therefore we conclude that $\mu = 1$ and $\sigma^2 = 1$.

(b) If $(X_t)_{t\geq 0}$ is a Brownian motion with drift μ and variance σ^2, then the process $(Z_t)_{t\geq 0}$ given by $Z_t = e^{X_t}$ is a geometric Brownian motion with drift α, where

$$\alpha = \mu + \sigma^2/2 = 3/2. \tag{26}$$

Denote $T := \min\{t : Z_t = 2 \text{ or } Z_t = 1/2\}$. Compute

$$1 - \frac{2\alpha}{\sigma^2} = 1 - \frac{3}{1} = -2. \tag{27}$$

Using the "gambler's ruin" theorem for geometric Brownian motion (Lecture 27, page 13)

$$P(Z_T = 2) = \frac{1 - (1/2)^{-2}}{2^{-2} - (1/2)^{-2}} = \frac{1 - 4}{1/4 - 4} = \frac{4}{5}. \tag{28}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

7. The fluctuations of the cash assets of a certain company are modeled by a Brownian motion with variance parameter $\sigma^2 = 2$ reflected at 0 (taking only positive values). Suppose that initially (at time $t = 0$) the cash assets of the company are equal to 10.

   Determine the probability that **at time** $t = 50$ the cash assets do not exceed 20. [Express the answer in terms of the CDF of the standard normal distribution $\Phi(x)$.]

   Solution.

   Denote by R_t the value of the cash assets at time t. Then by definition

$$R_t = |10 + \sqrt{2}B_t|, \tag{29}$$

   where $(B_t)_{t\geq 0}$ is a standard Brownian motion. Using this we compute

$$\begin{aligned} P(R_{50} \leq 20) &= P(|10 + \sqrt{2}B_{50}| \leq 20) && (30)\\ &= P(-30 \leq 10B_1 \leq 10) && (31)\\ &= P(-3 \leq B_1 \leq 1) && (32)\\ &= \Phi(1) - (1 - \Phi(3)) && (33)\\ &= 0.8413 - 0.00135 \approx 0.84 && (34) \end{aligned}$$

(ADDITIONAL SPACE FOR WORK, clearly INDICATE the problem you are working on)

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